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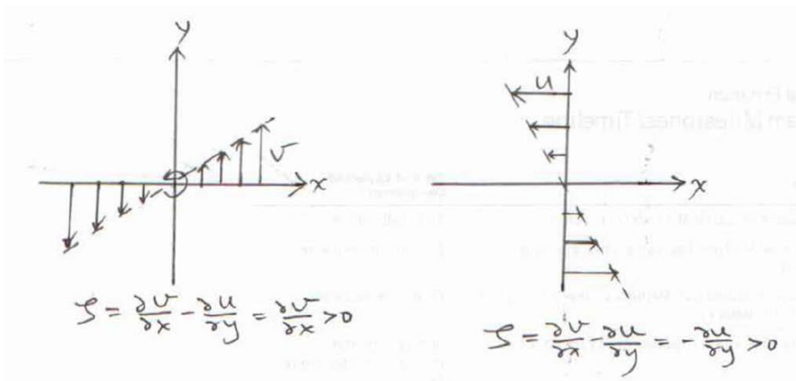
Chapter 5 Circulation and Vorticity

5.2 Vorticity

- Vorticity is a microscopic measure of rotation in a fluid.
- Based on the following flow patterns,



The vertical relative vorticity (ζ) can be defined: $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

and the vertical absolute vorticity (ζ_a) can be derived:

$$\zeta_a = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + f \; .$$

- The above relative and absolute vorticities can be extended to 3D vorticity and absolute vorticity

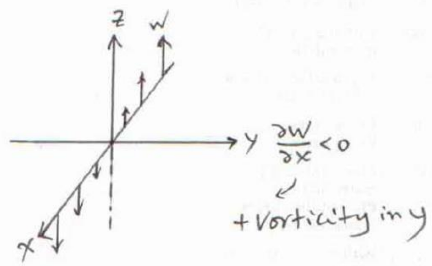
$$\boldsymbol{\omega} = \nabla \mathrm{x} \boldsymbol{V} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\boldsymbol{k} \; ,$$

$$\boldsymbol{\omega_a} = \nabla \mathrm{x} \boldsymbol{V_a} = \boldsymbol{\omega} + \nabla \mathrm{x} \boldsymbol{V_e} \; ,$$

$\zeta = \boldsymbol{k} \cdot \boldsymbol{\omega} = \boldsymbol{k} \cdot (\nabla \text{x} \boldsymbol{V})$,

$\zeta_a = \boldsymbol{k} \cdot \boldsymbol{\omega_a} = \boldsymbol{k} \cdot (\nabla \text{x} \boldsymbol{V}) + \boldsymbol{k} \cdot (\nabla \text{x} \boldsymbol{V_e}) = \boldsymbol{k} \cdot (\nabla \text{x} \boldsymbol{V}) + f$.

e.g.



- **Relation between ζ and C:**
  Applying the Stokes' theorem to the definition of circulation, we may obtain the relation between vorticity and circulation:

$$C = \oint \boldsymbol{V} \cdot \boldsymbol{dl} = \iint_A (\nabla \text{x} \boldsymbol{V}) \cdot \boldsymbol{n} dA = \iint_A \zeta \, dA = \bar{\zeta} A \qquad (4.8)'$$

Stokes' Theorem: Stokes' theorem (e.g. see Adv. Calculus for appl. by Hildebrand) link contour integration to area integration,

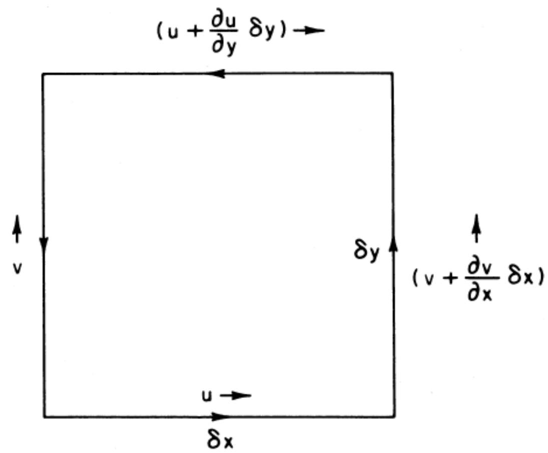
$$\oint \boldsymbol{V} \cdot \boldsymbol{dl} = \iint_A (\nabla \text{x} \boldsymbol{V}) \cdot \boldsymbol{n} dA ,$$

where A is the surface area enclosed by the contour for contour integration and $\boldsymbol{n}$ is a unit vector perpendicular to the surface in counterclockwise sense.

The above relation can also be obtained by evaluating the circulation along each side of a small rectangle:

$$\delta C \equiv \oint \boldsymbol{V} \cdot \boldsymbol{dl} = u\delta x + \left(v + \frac{\partial v}{\partial x} \delta x \right) \delta y - \left(u + \frac{\partial u}{\partial y} \delta y \right) \delta x - v \delta y$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \delta x \delta y = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \delta A$$



Thus, we have

$$\frac{\delta C}{\delta A} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

This implies that

$$C \equiv \oint \boldsymbol{V} \cdot \boldsymbol{dl} = \overline{\zeta} A \qquad (4.8)''$$

In words, circulation is roughly equal to the mean vorticity times the area enclosed by the integration contour.

The above equation may also be rewritten as

$$\frac{DC}{DA} = \zeta \qquad (4.8)$$

- **Vorticity in Natural Coordinates**

Definition of natural coordinates, (***t***, ***n***): ***t*** is a unit vector tangential to the local velocity vector, ***n*** is a unit vector perpendicular to ***t*** pointing to the left.

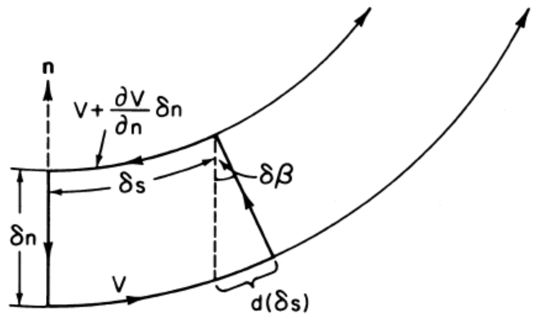


Fig. 4.5 Circulation for an infinitesimal loop in the natural coordinate system.

However, from Fig. 4.5, $d(\delta s) = \delta\beta\,\delta n$, where $\delta\beta$ is the angular change in the wind direction in the distance δs. Hence,

$$\delta C = \left(-\frac{\partial V}{\partial n} + V\frac{\delta\beta}{\delta s} \right) \delta n\ \delta s$$

or, in the limit $\delta n, \delta s \to 0$

$$\zeta = \lim_{\delta n, \delta s \to 0} \frac{\delta C}{(\delta n\ \delta s)} = -\frac{\partial V}{\partial n} + \frac{V}{R_s} \qquad (4.9)$$

where R_s is the radius of local curvature.

In the natural coordinate, the vertical vorticity is composed by the curvature vorticity (V/R) and shear vorticity ($-\partial V/\partial n$),

$$\zeta = \frac{V}{R} - \frac{\partial V}{\partial n},$$

where R is the radius of local curvature.

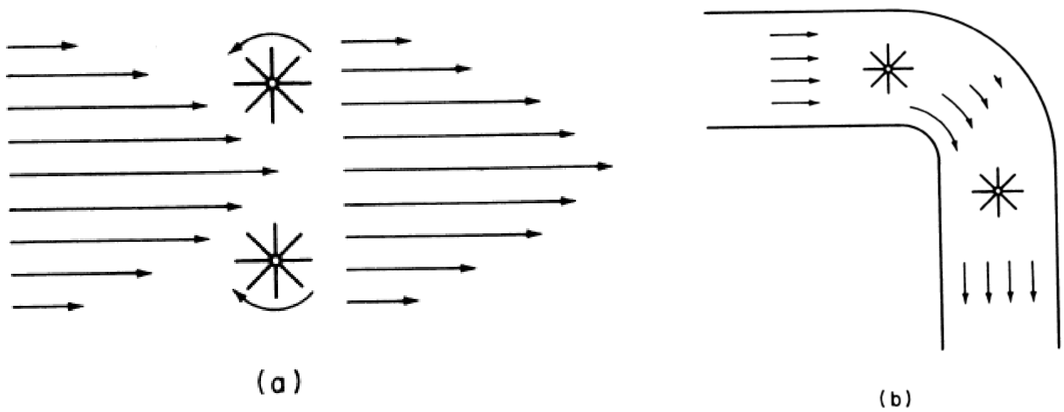


Fig. 4.6 Two types of 2-D flow with: (a) shear vorticity, and (b) curvature vorticity.