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Chapter 5 Circulation and Vorticity

5.0 Introduction

(Equation editor: $D/Dt = \partial/\partial t + u\partial/\partial x$)

- Linear motion Circular motion

$u = d/T$ (linear velocity) $\omega = \alpha/T$ (angular velocity)

There are alternative ways to measure circular motion than the angular velocity.

- Two primary measures of rotation in a fluid are:
  - Circulation – macroscopic measure for a fluid area, which is a scalar integral quantity.
  - Vorticity – microscopic measure at a point of the fluid, which is a vector.

These quantities also allow us to apply the conservation of angular momentum to the fluid motion in an easier fashion.

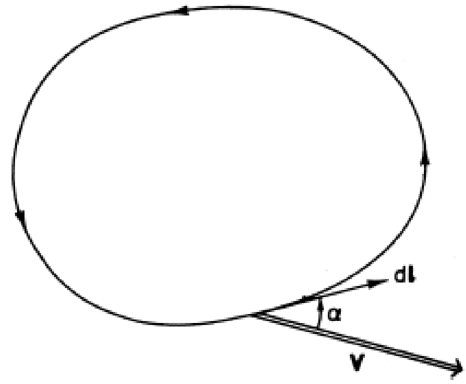
5.1 The Circulation Theorem

- Definition of circulation

The circulation, C, about a closed contour in a fluid is defined as the line integral evaluated counterclockwise along the contour of the component of the velocity vector that is locally tangent to the contour:

$$C \equiv \oint \boldsymbol{V} \cdot \boldsymbol{dl} = \oint V|dl|\cos\alpha \ .$$

Fig. 4.1 Circulation about a closed contour.



Since

$$C = \oint \boldsymbol{V} \cdot \boldsymbol{dl} = \oint V|dl|\cos\alpha \ ,$$

$C > 0$ for cyclonic flow.

- Claim: Circulation is twice of the angular velocity timing area (2Ω x Area) for a disc of fluid in a solid body rotation.

  Proof: Consider a solid-body rotation.

  V and ***dl*** are in the same direction all the time => $\alpha = 0$, i.e. cos α = 1. This gives

  $$C = \oint \boldsymbol{V} \cdot \boldsymbol{dl} = \oint V dl = \int_0^{2\pi} (\Omega r)(r d\alpha) = \int_0^{2\pi} \Omega r^2 d\alpha = \Omega r^2 \int_0^{2\pi} d\alpha = 2\pi\Omega r^2 \ .$$

  Thus, $$\frac{C}{\pi r^2} = 2\Omega \ .$$

  That is: $$\frac{Circulation}{Area} = Twice\ of\ the\ angular\ velocity$$

- Taking the line integral of Newton's second law for a closed chain of fluid particles, with the help of Stokes' theorem, leads to the circulation theorem.

- Circulation Theorem

Recall Eq. (2.8) in Ch. 2 of Holton (2004),

$$\frac{DV}{Dt} = -2\Omega \, \mathrm{x} \, \boldsymbol{V} - \frac{1}{\rho}\nabla p - g\boldsymbol{k} \quad . \tag{2.8}$$

The above equation can be rewritten as

$$\frac{D_a \boldsymbol{V_a}}{Dt} = -\frac{1}{\rho}\nabla p - \nabla \phi \, , \tag{2.8)'$$

because $\nabla \phi = \frac{\partial (gz)}{\partial x}\boldsymbol{i} + \frac{\partial (gz)}{\partial y}\boldsymbol{j} + \frac{\partial (gz)}{\partial z}\boldsymbol{k} = \frac{\partial (gz)}{\partial z}\boldsymbol{k} = g\boldsymbol{k}$.

Taking $\cdot d\boldsymbol{l}$ on both sides of (2.8)' and a close line integral lead to

$$\oint \frac{D_a \boldsymbol{V_a}}{Dt} \cdot d\boldsymbol{l} = -\oint \frac{1}{\rho}\nabla p \cdot d\boldsymbol{l} - \oint \nabla \phi \cdot d\boldsymbol{l} \quad . \tag{4.1}$$

The integrand of the left-hand side can be rewritten as

$$\frac{D_a \boldsymbol{V_a}}{Dt} \cdot d\boldsymbol{l} = \frac{D}{Dt}\left(\boldsymbol{V_a} \cdot d\boldsymbol{l}\right) - \boldsymbol{V_a} \cdot \frac{D_a}{Dt}(d\boldsymbol{l}) \quad .$$

Since $\boldsymbol{l}$ is a position vector, we have

$$\frac{D_a \boldsymbol{l}}{Dt} = \boldsymbol{V_a} \, ,$$

$$\frac{D_a \boldsymbol{V_a}}{Dt} \cdot d\boldsymbol{l} = \frac{D}{Dt}\left(\boldsymbol{V_a} \cdot d\boldsymbol{l}\right) - \boldsymbol{V_a} \cdot d\boldsymbol{V_a} \tag{4.2}$$

Substituting (4.2) into (2.8)'' leads to

$$\frac{D}{Dt}(\boldsymbol{V_a} \cdot d\boldsymbol{l}) - \boldsymbol{V_a} \cdot d\boldsymbol{V_a} = -\frac{1}{\rho}\nabla p \cdot d\boldsymbol{l} - \nabla\phi \cdot d\boldsymbol{l}.$$

Taking a close line integral of the above equation gives

$$\oint \frac{D}{Dt}(\boldsymbol{V_a} \cdot d\boldsymbol{l}) - \oint \boldsymbol{V_a} \cdot d\boldsymbol{V_a} = -\oint \frac{\nabla p \cdot d\boldsymbol{l}}{\rho} - \oint \nabla\phi \cdot d\boldsymbol{l}$$

(1) (2) (3) (4)

Term (1): $\oint \frac{D}{Dt}(\boldsymbol{V_a} \cdot d\boldsymbol{l}) = \frac{D}{Dt}\oint(\boldsymbol{V_a} \cdot d\boldsymbol{l}) = \frac{DC_a}{Dt}$

Term (2): $-\oint \boldsymbol{V_a} \cdot d\boldsymbol{V_a} = -\frac{1}{2}\oint d(\boldsymbol{V_a} \cdot \boldsymbol{V_a}) = 0$

(Because closed line integral of an exact differential is 0.)

Term (3): $-\oint \frac{\nabla p \cdot d\boldsymbol{l}}{\rho} = -\oint \frac{dp}{\rho}$

Term (4): $-\oint \nabla\phi \cdot d\boldsymbol{l} = -\oint d\phi = 0$.

(Again, closed line integral of an exact differential is 0.)

Thus, we obtain the circulation theorem:

$$\frac{DC_a}{Dt} = -\oint \frac{dp}{\rho} \quad (4.3)$$

The term on the right-hand side is called "solenoidal term". The physical meaning of the solenoidal term will be explained later.

[Kelvin's Circulation Theorem]

For a barotropic fluid, $\rho = \rho(p,T) = \rho(p)$, there is no temperature difference on an isobaric (pressure) surface. This lead to

$$\frac{DC_a}{Dt} = -\oint \frac{dp}{\rho} = 0 .$$

e.g., suppose $\rho = \rho(p) = ap$, where a is a constant, then

$$\oint \frac{dp}{\rho} = \oint \frac{dp}{ap} = \frac{1}{a}\oint \frac{dp}{p} = \frac{1}{a}\oint d(\ln p) = \frac{1}{a}\ln p\Big|_{p_o}^{p_o} = 0 .$$

(Note that the closed line integral of an exact differential is always 0.)

In other words, in a barotropic atmosphere or fluid in general, the absolute circulation is conserved following the motion, i.e.

$$\frac{DC_a}{Dt} = 0 .$$

This is called the Kelvin's circulation theorem.

It can be shown that Kelvin's circulation theorem is analogous to the conservation of angular momentum.

Recall that

Linear momentum: $P_{linear} = mv$

Angular momentum: $L = I\Omega$ where I is the moment of Inertia, which depends on the shape of the object, and Ω is the angular velocity.

e.g., $I = (1/2)\, MR^2$ for a disk, where M is the mass and R is the radius of the disk rotating about an axis perpendicular to its plane passing through its center.

$I = MR^2$ for a ring, where M is the mass and R is the radius of the ring.

- For meteorological applications, it is more convenient to use relative circulation, instead of absolute circulation as adopted in Kelvin's circulation theorem.

- Bjerknes extends Kelvin circulation theorem to the "Bjerknes circulation theorem".

  Recall from Holton's Eq. (2.5)

$$\boldsymbol{V}_a = \boldsymbol{V} + \boldsymbol{\Omega} \times \boldsymbol{r} \qquad \text{H(2.5)}$$

  where $\boldsymbol{r}$ is the position vector from the earth's center. *The physical meaning of H(2.5) can be illustrated by considering a person standing on the rim of a Mary-go-around throws a ball into the center.*

  Taking $\cdot\, \boldsymbol{dl}$ of the above equation and integrate along a closed contour on earth's surface gives

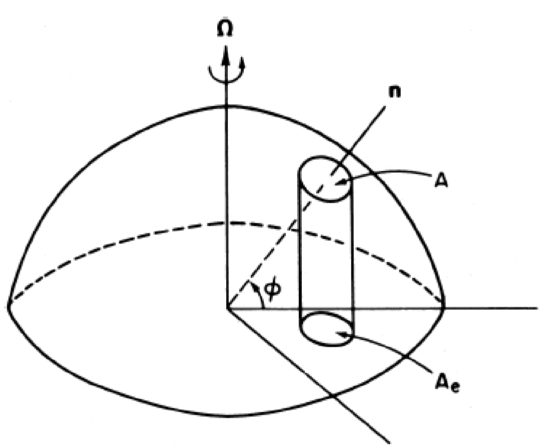
$$\oint \boldsymbol{V}_a \cdot \boldsymbol{dl} = \oint \boldsymbol{V} \cdot \boldsymbol{dl} + \oint (\boldsymbol{\Omega} \times \boldsymbol{r}) \cdot \boldsymbol{dl}$$

  Absolute Circulation   Relative Circulation

  After some manipulation of the second term on the right-hand side, the above equation can be rewritten as

$$C_a = C + 2\Omega A \sin\overline{\phi} = C + 2\Omega A_e, \qquad (4.4)$$

  Here A_e = $A_e = A\sin\overline{\phi}$ is the projection of A on equatorial plane as shown below:



Taking integration of the above equation involving C_a yields

$$\frac{DC_a}{Dt} = \frac{DC}{Dt} + 2\Omega \frac{D(A \sin \bar{\phi})}{Dt}, \text{ or}$$

$$\frac{DC}{Dt} = \frac{DC_a}{Dt} - 2\Omega \frac{D(A \sin \bar{\phi})}{Dt}.$$

Inserting the circulation theorem into the above equation gives the Bjerknes circulation theorem:

$$\frac{DC}{Dt} = -\oint \frac{dp}{\rho} - 2\Omega \frac{D}{Dt}(A \sin \bar{\phi}) \tag{4.5}$$

For a barotropic atmosphere (no temperature variation on an isobaric surface), Eq. (4.5) reduces to

$$\frac{DC}{Dt} = -2\Omega \frac{D}{Dt}(A \sin \bar{\phi}).$$

Integrating the above equation from time 1 to 2 leads to

$$\int_1^2 \frac{DC}{Dt} dt = -2\Omega \int_1^2 \frac{D}{Dt}(A \sin \bar{\phi}) dt, \text{ or}$$

$$C_2 - C_1 = -2\Omega (A_2 \sin \bar{\phi}_2 - A_1 \sin \bar{\phi}_1). \tag{4.6}$$

That is, circulation changes if the area of the fluid chain or the latitude changes.

- Applications of Bjerknes circulation theorem

Example 1: Consider a circular region of area A, originally located at equator with no circulation, which is moved to North Pole with area conserved (i.e. A = constant). Estimate the final circulation C.

Then, $C_2 - C_1 = -2\Omega(A_2 \sin\overline{\phi}_2 - A_1 \sin\overline{\phi}_1)$ becomes

$$C_2 = -2\Omega(A\sin(\pi/2) - A\sin(0)) = -2\Omega A = -2\pi r^2 \Omega$$

$$V_2 = \frac{C}{2\pi r} = -\Omega r$$

For r = 100 km, $V_2 = -\Omega r = -(7.292x10^{-5} s^{-1})(10^5 m) \approx -7.3 ms^{-1}$.

Example 2: Sea-breeze circulation [Reading Assignment]

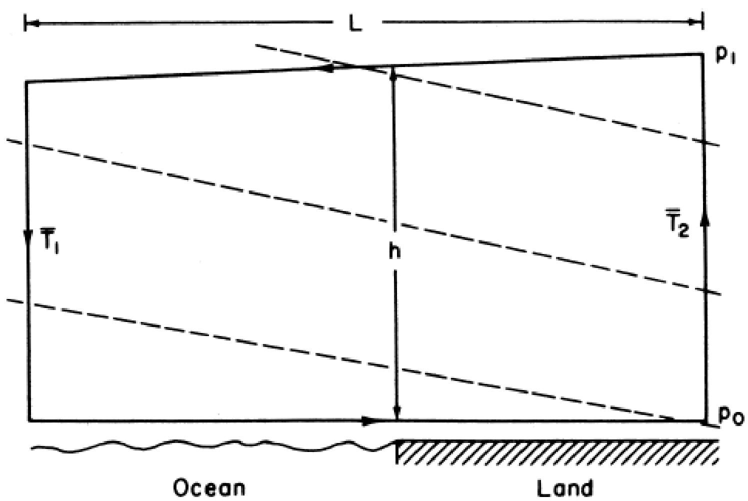


Fig. 4.3 Application of the circulation theorem to the sea breeze problem. The closed heavy solid line is the loop about which the circulation is to be evaluated. Dashed lines indicate surfaces of constant density.

Note that the constant density surfaces are tilted in opposite way of the density surface (i.e., from low to high values).

$$\frac{DC_a}{Dt} = -\oint \frac{dp}{\rho} = -\oint RT \frac{dp}{p} = -\oint RT\, d(\ln p)$$

$$= -\int_a^b RT\, d(\ln p) - \int_b^c RT\, d(\ln p) - \int_c^d RT\, d(\ln p) - \int_d^a RT\, d(\ln p)$$

Where *a, b, c, and d* denote the lower left, lower right, upper right corner, and upper left corners, respectively.

Assuming the isobaric (pressure) surface is nearly horizontal, then the 1st and 3rd terms are approximately 0.

$$\frac{DC_a}{Dt} = -\int_b^c RT\,d(\ln p) - \int_d^a RT\,d(\ln p) = -R\overline{T}_2\int_b^c d(\ln p) - R\overline{T}_1\int_d^a d(\ln p)$$

$$\frac{DC_a}{Dt} = -R\overline{T}_2 \ln\frac{p_c}{p_b} - R\overline{T}_1 \ln\frac{p_a}{p_d} = R(\overline{T}_2 - \overline{T}_1)\ln\frac{p_o}{p_1}.$$

- Physical meaning of the solenoidal term in Holton's Eq. (4.3) and its application to sea-breeze circulation.