AST 851/ASME4 433 Dynamics of Atmosphere Dr. Yuh-Lang Lin
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Chapter 4
Elementary Applications of the Basic Equations

4.1 Basic Equations in Isobaric Coordinates

(Ref: Holton Sec. 3.1)

➢ The Horizontal Momentum Equation

The approximate horizontal momentum equations (2.24) and (2.25) may be written in vectoral form as

$$\frac{Du}{Dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} \tag{2.24}$$

$$\frac{Dv}{Dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y}\,. \tag{2.25}$$

Inertial Force   Coriolis   PGF

[1st term: Also called total rate of change following the motion; total derivative, material derivative]

$$\frac{D\boldsymbol{V}}{Dt} + f\,\boldsymbol{k}\,x\,\boldsymbol{V} = -\frac{1}{\rho}\nabla p \tag{3.1}$$

where $\boldsymbol{V} = u\,\boldsymbol{i} + v\,\boldsymbol{j}$ is the horizontal velocity vector.

Substituting the following gradient force in isobaric coordinates,

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\right)_z = -\left(\frac{\partial \phi}{\partial x}\right)_p, \tag{1.20}$$

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial y}\right)_z = -\left(\frac{\partial \phi}{\partial y}\right)_p \tag{1.21}$$

into (3.1) leads to

$$\frac{D\boldsymbol{V}}{Dt} + f\,\boldsymbol{k}\,x\boldsymbol{V} = -\nabla_p \phi \qquad (3.2)$$

where ∇_p is the horizontal gradient operator applied with pressure held constant.

Because p is the independent vertical coordinate, we must expand the total derivative as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{Dx}{Dt}\frac{\partial}{\partial x} + \frac{Dy}{Dt}\frac{\partial}{\partial y} + \frac{Dp}{Dt}\frac{\partial}{\partial p} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + \omega\frac{\partial}{\partial p} \qquad (3.3)$$

Here $\omega = Dp/Dt$ is called the *omega vertical motion* which is defined as the pressure change following the motion, equivalent to $w = Dz/Dt$ in height coordinates.

Note that for synoptic motions, $\omega \approx -\rho g w$.

- From (3.2), the geostrophic relation in isobaric coordinates can be written as

$$f\boldsymbol{V_g} = \boldsymbol{k} \,\mathrm{x}\, \nabla_p \phi \qquad (3.4)$$

or in scalar form

$$fu_g = -\frac{\partial \phi}{\partial y}, \qquad (3.4a)$$

$$fv_g = \frac{\partial \phi}{\partial x}. \qquad (3.4b)$$

Note there is no density present in (3.4).

In addition, on an f-plane (i.e., f is constant), we have

$$\nabla_p \cdot \boldsymbol{V_g} = 0$$

That is, there is no divergence for the geostrophic flow (non-divergent).

- The continuity equation in the isobaric coordinates can be derived directly from Eq. (2.31)

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \boldsymbol{V} = 0, \qquad (2.31)$$

But it is easier to derive the isobaric form by considering a Lagrangian control volume $\delta V = \delta x \delta y \delta z$ and $\delta p = -\rho g \delta z$. The mass, $\delta M = \rho \delta V = -\, \delta x \delta y \delta p / g$, is conserved following the motion,

$$\frac{1}{\delta M}\frac{D}{Dt}\delta M = \frac{g}{\delta x \delta y \delta p}\frac{D}{Dt}\left(\frac{\delta x \delta y \delta p}{g}\right) = 0.$$

Applying the chain rule, we obtain

$$\frac{1}{\delta x}\delta\left(\frac{Dx}{Dt}\right) + \frac{1}{\delta y}\delta\left(\frac{Dy}{Dt}\right) + \frac{1}{\delta p}\delta\left(\frac{Dp}{Dt}\right) = 0$$

which gives us

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)_p + \frac{\partial \omega}{\partial p} = 0. \qquad (3.5)$$

- The thermodynamic energy equation

Taking the total derivative of the equation of state

$$p\alpha = RT \tag{a}$$

Gives

$$p\frac{D\alpha}{Dt} + \alpha\frac{Dp}{Dt} = R\frac{DT}{Dt} \tag{b}$$

Now consider the first law of thermodynamics

$$du + dw = dq$$

or

$$c_v dT + p d\alpha = dq \tag{c}$$

Since $c_p = c_v + R$, (c) can be rewritten as

$$c_p dT - \alpha dp = dq \tag{d}$$

Taking total derivative of (d) gives

$$c_p\frac{DT}{Dt} - \alpha\frac{Dp}{Dt} = J \tag{2.42}$$

where $J = Dq/Dt$ is the diabatic heating rate ($J\ kg^{-1}\ s^{-1}$).
Equation (2.42) may be rewritten as

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} - S_p\omega = \frac{J}{c_p} \tag{3.6}$$

where $J = \frac{Dq}{Dt}$ is the diabatic heating rate and

$$S_p \equiv \frac{RT}{c_p p} - \frac{\partial T}{\partial p} = -\frac{T}{\theta}\frac{\partial \theta}{\partial p}. \tag{3.7}$$

or

$$S_p \equiv \frac{\Gamma_d - \Gamma}{\rho g}$$

is a "static stability parameter".