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# Chapter 4 The Vorticity Equation

- Derivation of the vorticity equation

Cross-differentiating the zonal and meridional component equations with respect to *x* and *y* gives:

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}\right) \tag{4.10}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}\right) \tag{4.11}$$

Subtracting (4.10) from (4.11) and recalling that $\zeta=\partial v/\partial x-\partial u/\partial y$ lead to the vorticity equation:

$$\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}+w\frac{\partial \zeta}{\partial z}+(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+v\frac{\partial f}{\partial y}=\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) \tag{4.12}$$

Since $\frac{Df}{Dt}=v\frac{\partial f}{\partial y}$, (4.12) can be rewritten as

$$\frac{\partial \zeta}{\partial t}=-V\cdot\nabla(\zeta+f)-(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

- Physical meaning of individual terms of the vorticity equation Eq. (4.17) may also be written as

$$\frac{\partial \zeta}{\partial t} = -\boldsymbol{V}\cdot\nabla(\zeta + f) - (\zeta + f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

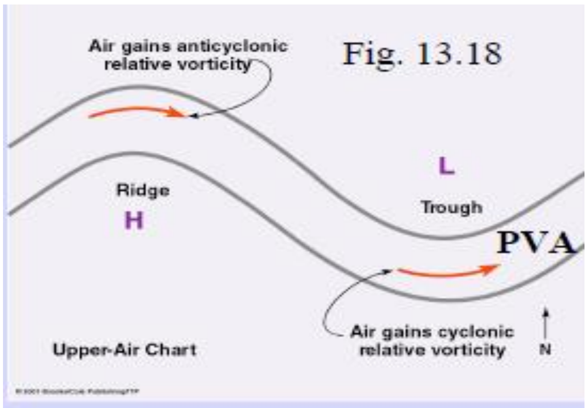
Local rate of change of vertical vorticity — Absolute vorticity advection — Divergence (stretching) term

$$-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

Tilting (twisting) term — Solenoidal term

- Physical interpretation of the vorticity advection term

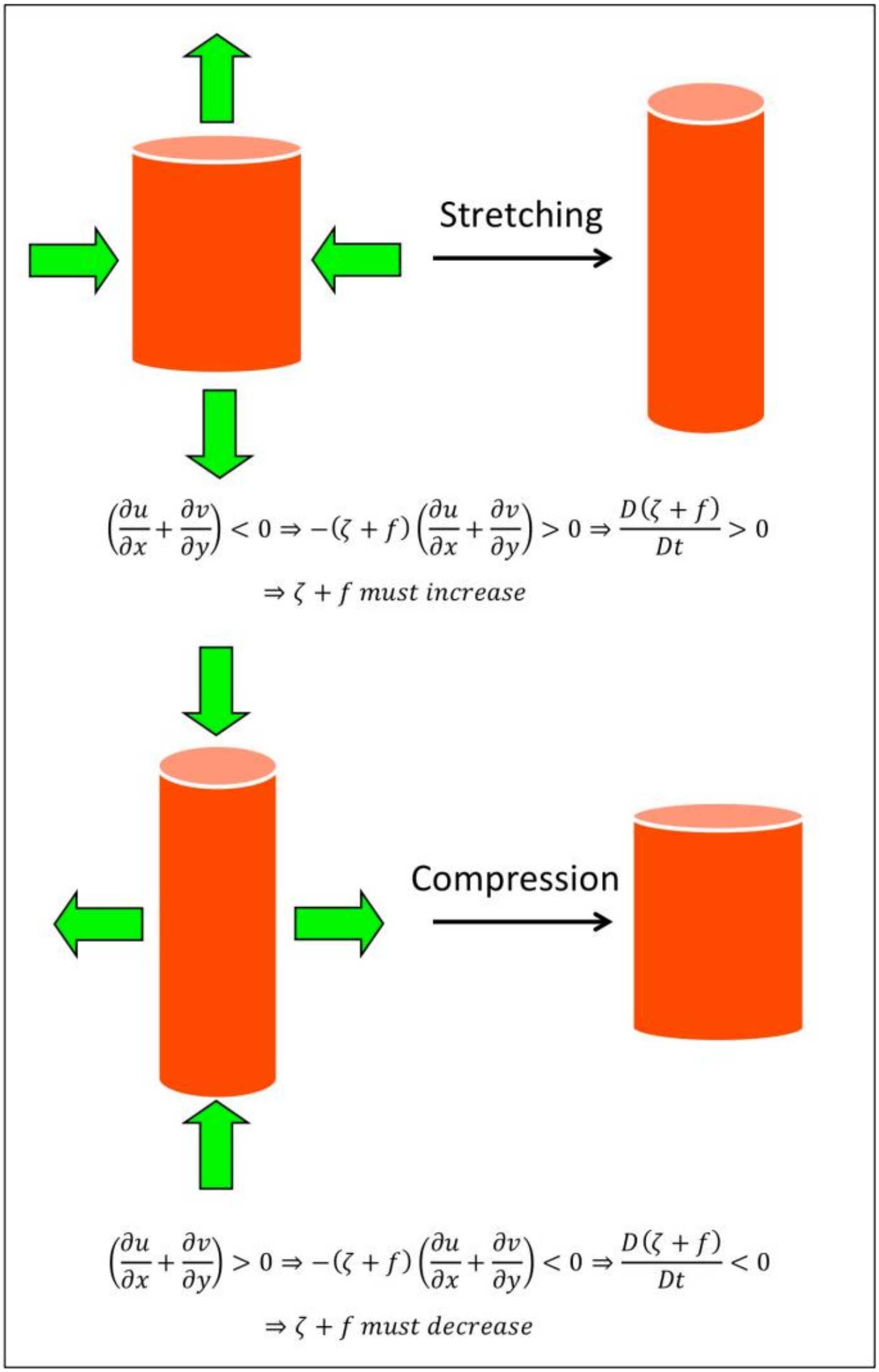
$$-V\cdot\nabla(\zeta+f) = \left(-u\frac{\partial\zeta}{\partial x} - v\frac{\partial\zeta}{\partial y}\right) - \beta v \qquad \text{(where } \beta = \partial\zeta/\partial y\text{)}$$



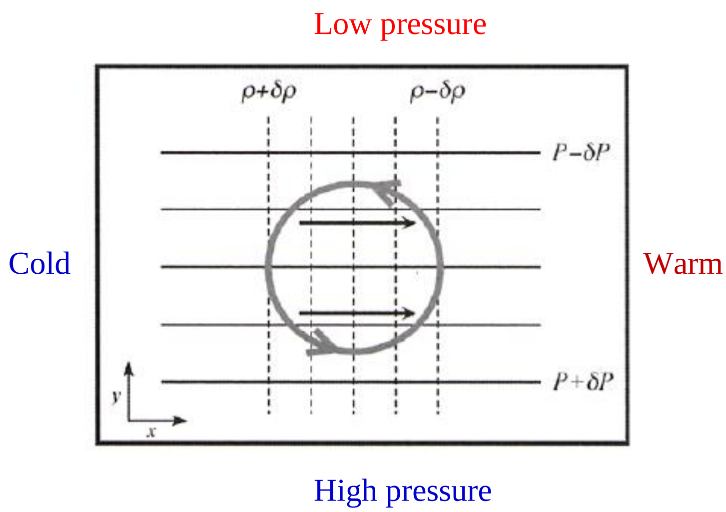

$\frac{\partial\zeta}{\partial t} \propto -u\frac{\partial\zeta}{\partial x}$: $<0$ $>0$

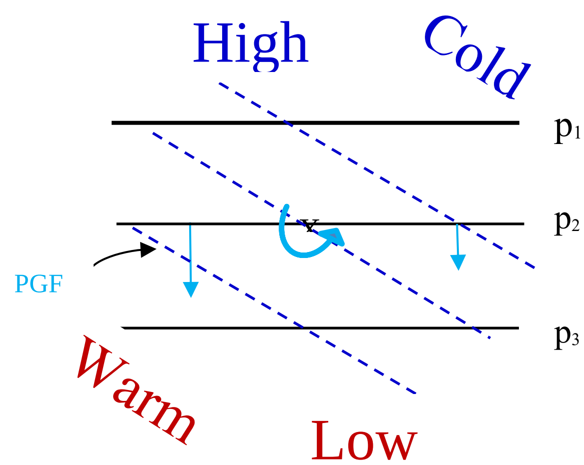
$\frac{\partial\zeta}{\partial t} \propto -\beta v$: $>0$ $<0$

- Physical interpretation of the divergence (stretching) term



- Physical interpretation of the solenoidal term (also see the section of sea-breeze circulation)




Dashed lines: isotherms

$$\boxed{\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \cancel{\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}}\right)} > 0$$

$+ \quad - \quad -$

Thus, the solenoidal term will induce increase of $\zeta$.

- Physical interpretation of the tilting term

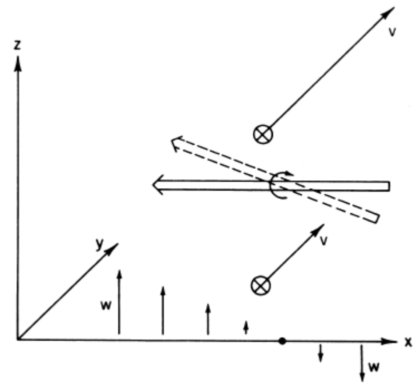



**Fig. 4.12** Vorticity generation by the tilting of a horizontal vorticity vector (double arrow).

## Application to the formation of mesocyclones and tornadoes

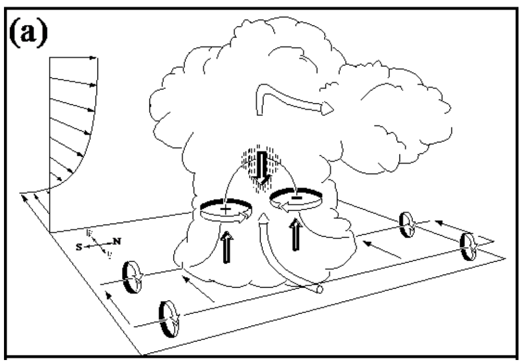
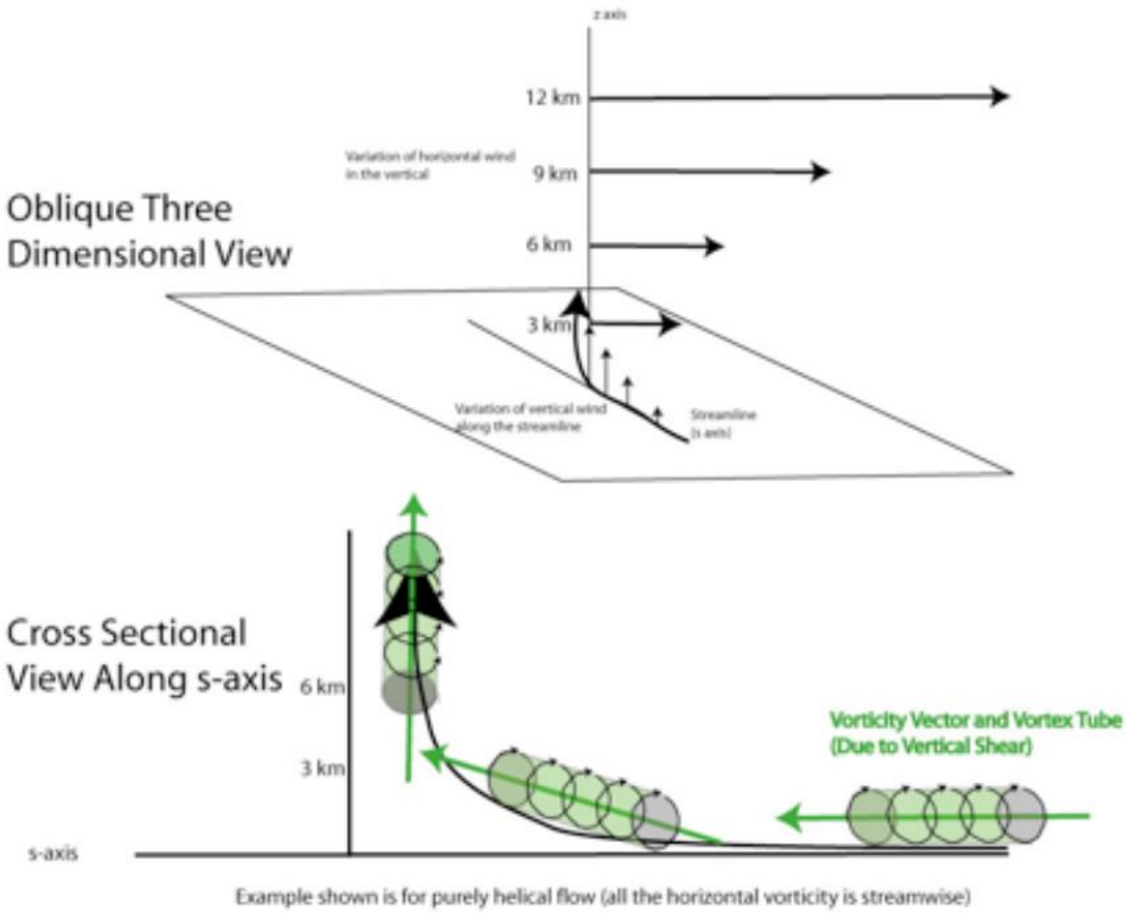



Fig. 8.20: A schematic depicting rotation development and the storm splitting. (a) Rotation development: In the early stage, a pair of vortices forms through tilting of horizontal vorticity associated with the (westerly) environmental shear. (b) Storm splitting: In the later stage, the precipitation induced downdraft splits the updraft. At this stage, vortex lines are tilted downward, producing two vortex pairs. Cylindrical arrows denote the direction of the storm-relative airflow, and heavy solid lines represent vortex lines with the sense of rotation denoted by circular arrows. Shaded arrows represent the forcing promoting new updraft and downdraft acceleration. Vertical dashed lines denote regions of precipitation. Frontal symbols at the surface mark the boundary of cold air outflow. (Lin 2007, After Klemp)

# A mesocyclone may be developed by tilting of horizontal vorticity into the vertical



- Helicity

When the vertical vorticity is in phase with the vertical motion, it tends to develop into its maximum strength, thus lead to a ***tornoado genesis***. This can be diagnosed by helicity.

**A. Relationship of Three Dimensional Helicity to Three Dimensional Vorticity**

***(Note: Inclass Exercises Completed As Lab Session at Conclusion of the Discussion)***

The tendency of the atmosphere to have "helical" flow can be measured by computation of the "helicity." To understand helicity, imagine an air parcel having horizontal vorticity, that is, a spin around the y axis (but keep in mind that helicity has components on each of the three coordinate axes.).

Let's also say that the there is only a south wind component, or u=w=0. Then, the combination of the south wind and the vorticity around the y-axis will yield a flow that is "helical", that is, still a southerly wind, but with air rotating around the y axis as it is moving.

The three dimensional helicity is a scalar.

$$H = (\nabla \times \mathrm{v}) \bullet \mathrm{v}$$

Equation (3.3.29, Vol I, Bluestein)

expanded out is:

$$H = u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - v\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \quad (1)$$

Note that three dimensional helicity is the product of the three wind components with the three components of vorticity. Also, note that the far right hand term is a product of the vertical velocity and the vertical relative vorticity. The units of helicity are m $s^{-2}$ and of Storm Relative Helicity $m^2s^{-2}$ or J $kg^{-1}$.

# Storm Relative Helicity

Observations show that what is important in a thunderstorm developing a rotating updraft in its midlevels is not so much the helicity ingested (as suggested by equation (6)), but the STORM RELATIVE helicity ingested. To understand this, consider the case in which there is only southerly flow (say, 15 m/s) in an environment of great vertical wind shear. Say that this southerly flow is approaching a developing thunderstorm updraft. Equations (4), (5) and (6) would return large values of horizontal helicity suggesting that the thunderstorm's updraft would develop cyclonic helicity.

However, suppose a thunderstorm develops and is moving northward at 15 m/s. In that case, the thunderstorm would never "feel" the helicity. This is the reason that, operationally, the STORM RELATIVE HELICITY is of most importance.

$$H = -\int_0^h \left[ (\vec{\mathrm{v}} - \vec{\mathrm{c}}) \bullet \vec{k} \times \left( \frac{\partial \vec{\mathrm{v}}}{\partial z} \right) \right] dz \quad \text{(9a)}$$

$$H = -\int_0^h \left[ -(u - c_x)\left( \frac{\partial v}{\partial z} \right) + (v - c_y)\left( \frac{\partial u}{\partial z} \right) \right] dz \quad \text{(9b)}$$

and for a purely south wind

$$H = -\int_0^h \left[ (v - c_y)\left( \frac{\partial u}{\partial z} \right) \right] dz \qquad \text{(9c)}$$

- Vorticity Equation in Isobaric (pressure) Coordinates

$$\frac{\partial \zeta}{\partial t} = -\boldsymbol{V}\cdot\nabla(\zeta + f) - \omega\frac{\partial \zeta}{\partial p} - (\zeta + f)\nabla\cdot\boldsymbol{V} + k\cdot\left(\frac{\partial \boldsymbol{V}}{\partial p}\,\mathrm{x}\,\nabla\omega\right) \qquad (4.21)$$

Note the solenoidal term is implicit.

- Scale analysis of the Vorticity Equation

Based on typical observed magnitudes for synoptic-scale motions, scales are chosen as follows:

| | |
|---|---|
| $U \sim 10\,\mathrm{m\,s^{-1}}$ | horizontal scale |
| $W \sim 1\,\mathrm{cm\,s^{-1}}$ | vertical scale |
| $L \sim 10^6\,\mathrm{m}$ | length scale |
| $H \sim 10^4\,\mathrm{m}$ | depth scale |
| $\delta p \sim 10\,\mathrm{hPa}$ | horizontal pressure scale |
| $\rho \sim 1\,\mathrm{kg\,m^{-3}}$ | mean density |
| $\delta\rho/\rho \sim 10^{-2}$ | fractional density fluctuation |
| $L/U \sim 10^5\,\mathrm{s}$ | time scale |
| $f_0 \sim 10^{-4}\,\mathrm{s^{-1}}$ | Coriolis parameter |
| $\beta \sim 10^{-11}\,\mathrm{m^{-1}\,s^{-1}}$ | "beta" parameter |

The vorticity equation may be approximated by

$$\frac{D_h(\zeta + f)}{Dt} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad (4.22b)$$

Where $D_h / Dt \equiv \partial/\partial t + u\partial/\partial x + v\partial/\partial y$ .