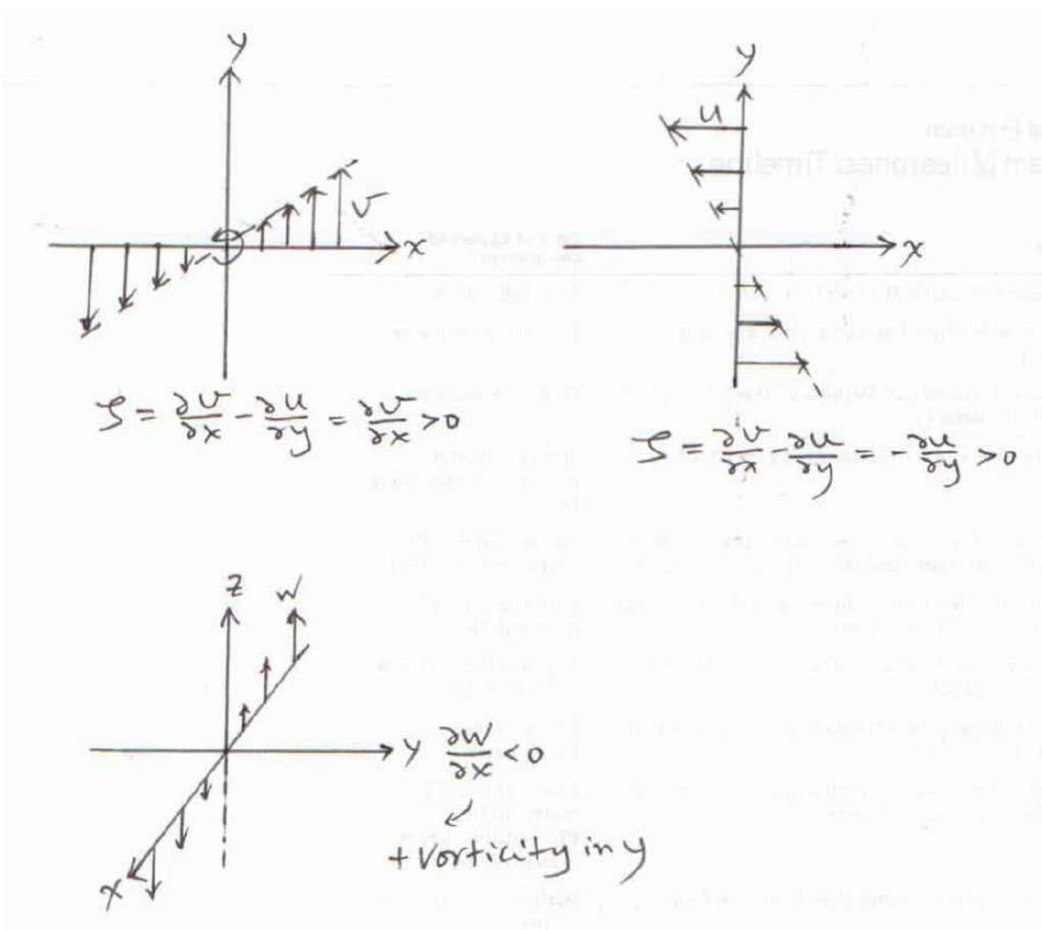
ASME 434 Atmospheric Dynamics II
Department of Physics
Applied Sci&Tech PhD Program
NC A&T State University

Dr. Yuh-Lang Lin
http://mesolab.org
ylin@ncat.edu

Chapter 3 Vorticity

(Holton Sec. 4.2 - Vorticity; Equation editor: $D/Dt = \partial/\partial t + u\partial/\partial x$)

- Vorticity is a microscopic measure of rotation in a fluid.



- Definitions of 3D vorticity and absolute vorticity

$$\boldsymbol{\omega} = \nabla \mathrm{x} \boldsymbol{V} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \boldsymbol{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \boldsymbol{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \boldsymbol{k} \, ,$$

$$\boldsymbol{\omega_a} = \nabla \mathrm{x} \boldsymbol{V_a} = \boldsymbol{\omega} + \nabla \mathrm{x} \boldsymbol{V_e} \, ,$$

$$\zeta = \boldsymbol{k} \cdot \boldsymbol{\omega} = \boldsymbol{k} \cdot (\nabla \mathrm{x} \boldsymbol{V}) \, ,$$

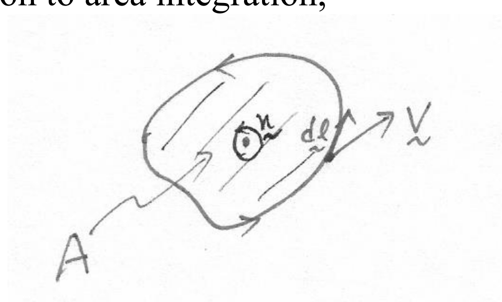
$$\zeta_a = \boldsymbol{k} \cdot \boldsymbol{\omega_a} = \boldsymbol{k} \cdot (\nabla \mathrm{x} \boldsymbol{V}) + \boldsymbol{k} \cdot (\nabla \mathrm{x} \boldsymbol{V_e}) = \boldsymbol{k} \cdot (\nabla \mathrm{x} \boldsymbol{V}) + f \, .$$

- Definitions of vertical relative vorticity (ζ) and vertical absolute vorticity:

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}; \qquad \zeta_a = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + f \, .$$

- **Relation between ζ and C:**
  Applying the Stokes' theorem to the definition of circulation, we may obtain the relation between vorticity and circulation:

Stokes' Theorem: Stokes' theorem (e.g. see Adv. Calculus for appl. by Hildebrand) links contour integration to area integration,



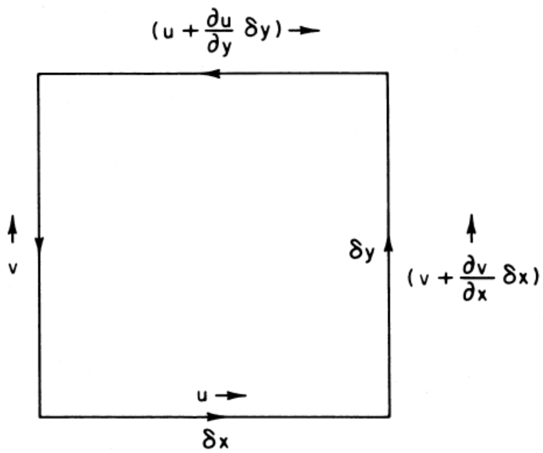
$$\oint \boldsymbol{V} \cdot \boldsymbol{dl} = \iint_A (\nabla \mathrm{x} \boldsymbol{V}) \cdot \boldsymbol{n} dA \, ,$$

where A is the surface area enclosed by the contour for contour integration and $\boldsymbol{n}$ is a unit vector perpendicular to the surface in counterclockwise sense.

The above relation can also be obtained by evaluating the circulation along each side of a small rectangle:

$$\delta C \equiv \oint \boldsymbol{V} \cdot \boldsymbol{dl} = u\delta x + \left(v + \frac{\partial v}{\partial x}\delta x \right)\delta y - \left(u + \frac{\partial u}{\partial y}\delta y \right)\delta x - v\delta y$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\delta x \delta y = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\delta A$$



Thus, we have

$$\frac{\delta C}{\delta A} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

This implies that

$$C \equiv \oint \boldsymbol{V} \cdot \boldsymbol{dl} = \overline{\zeta} A \qquad (4.8)\text{''}$$

In words, circulation is roughly equal to the mean vorticity times the area enclosed by the integration contour.

The above equation may also be rewritten as

$$\frac{DC}{DA} = \zeta \tag{4.8}$$

- **Vorticity in Natural Coordinates**

Definition of natural coordinates, (***t***, ***n***): ***t*** is a unit vector tangential to the local velocity vector, ***n*** is a unit vector perpendicular to ***t*** pointing to the left.

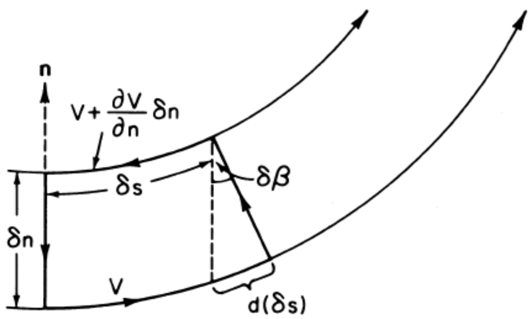


Fig. 4.5 Circulation for an infinitesimal loop in the natural coordinate system.

However, from Fig. 4.5, $d(\delta s) = \delta\beta\,\delta n$, where $\delta\beta$ is the angular change in the wind direction in the distance δs. Hence,

$$\delta C = \left(-\frac{\partial V}{\partial n} + V\frac{\delta\beta}{\delta s}\right)\delta n\ \delta s$$

or, in the limit $\delta n, \delta s \rightarrow 0$

$$\zeta = \lim_{\delta n, \delta s \to 0} \frac{\delta C}{(\delta n\ \delta s)} = -\frac{\partial V}{\partial n} + \frac{V}{R_s} \tag{4.9}$$

where R_s is the radius of local curvature.

In this coordinate system, the vertical vorticity is composed by the curvature vorticity (V/R) and shear vorticity ($-\partial V/\partial n$),

$$\zeta = \frac{V}{R} - \frac{\partial V}{\partial n},$$

where R is the radius of local curvature.

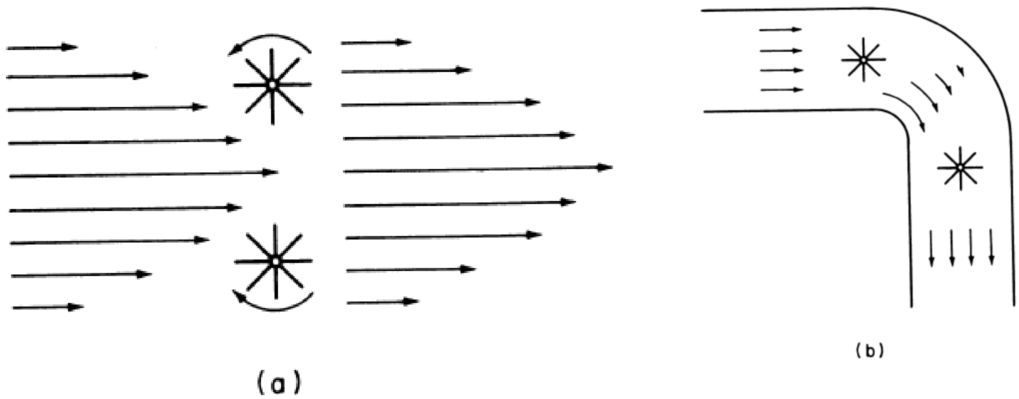
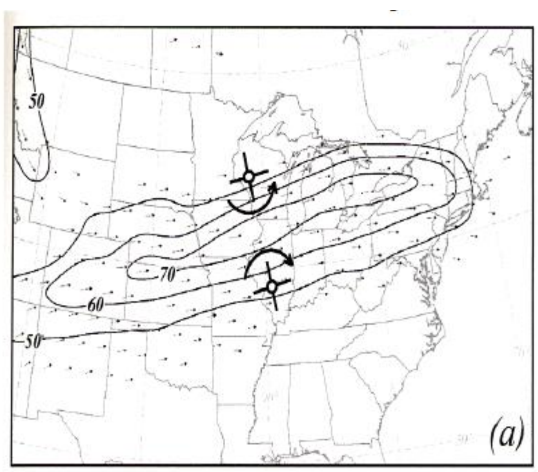


Fig. 4.6 Two types of 2-D flow with: (a) shear vorticity, and (b) curvature vorticity.

Example of shear vorticity and curvature vorticity:

300mb isotachs

Shear Vorticity



Even straight-line motion may have vorticity if the speed changes normal to the flow axis.

300mb geopotential heights

Curvature Vorticity