ASME 434 Atmospheric Dynamics II   Dr. Yuh-Lang Lin

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Chapter 2 Circulation Theorems

(Holton Sec. 4.1 - Circulation Theorems)

- Linear motion: measured by (linear) velocity

    $v = d/T$
    (velocity = distance traveled/time to travel distance)

    Circular motion: measured by angular velocity

    $\omega = \alpha / T$
    (angular velocity = angle traveled/time to travel angle)

    There are alternative ways to measure circular motion than the angular velocity.

- Two primary measures of rotation in a fluid are:

    ➢ Circulation – macroscopic measure for a fluid area, which is a scalar integral quantity.
    ➢ Vorticity – microscopic measure at a point of the fluid, which is a vector.

    These quantities also allow us to apply the conservation of angular momentum to the fluid motion in an easier fashion.

➢ Definition of Circulation

The circulation, C, about a closed contour in a fluid is defined as the line integral evaluated counterclockwise along the contour of the component of the velocity vector that is locally tangent to the contour:

$$C \equiv \oint V \cdot dl = \oint V |dl| \cos\alpha$$.

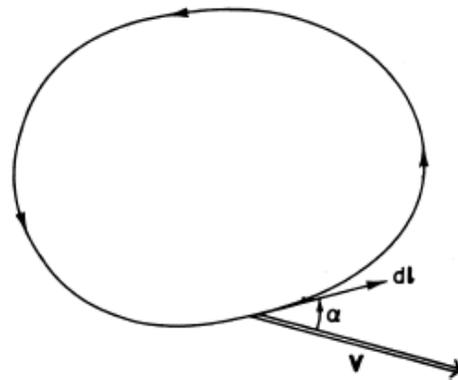**Fig. 4.1**   Circulation about a closed contour.

Since

$$C = \oint V \cdot dl = \oint V \cos\alpha \, dl ,$$

$C > 0$ for cyclonic flow.

• Claim: Circulation is twice of the angular velocity times area (2Ω x Area) for a disc of fluid in a solid body rotation.

Proof: Consider a solid-body rotation.

V and dl are in the same direction all the time => $\alpha = 0$, i.e. $\cos\alpha = 1$. This gives

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$$C = \oint \boldsymbol{V} \cdot \boldsymbol{dl} = \oint V dl = \int_0^{2\pi} (\Omega r)(rd\alpha) = \int_0^{2\pi} \Omega r^2 d\alpha = \Omega r^2 \int_0^{2\pi} d\alpha = 2\pi \Omega r^2.$$

Thus $\dfrac{C}{\pi r^2} = 2\Omega$.

That is: $\quad \dfrac{Circulation}{Area} = Twice\ of\ the\ angular\ velocity$

➢ The circulation theorem may be derived by taking the line integral of Newton's second law for a closed chain of fluid particles, with the help of Stoke's theorem.

2.1 Circulation Theorem

Recall Eq. (2.8) in Ch. 2 (Holton's),

$$\frac{D\boldsymbol{U}}{Dt} = -2\Omega \mathrm{x} \boldsymbol{U} - \frac{1}{\rho}\nabla p + \boldsymbol{g} + \boldsymbol{F}_r \qquad (2.8)$$

Let $\boldsymbol{U} = \boldsymbol{V}$ and assume friction is small, it leads to

$$\frac{D\boldsymbol{V}}{Dt} = -2\Omega \mathrm{x} \boldsymbol{V} - \frac{1}{\rho}\nabla p - g\boldsymbol{k}, \qquad (2.8a)$$

or

$$\frac{D_a \boldsymbol{V}_a}{Dt} = -\frac{1}{\rho}\nabla p - \nabla \phi, \qquad (2.8b)$$

because $\nabla \phi = \dfrac{\partial \phi}{\partial x}\boldsymbol{i} + \dfrac{\partial \phi}{\partial y}\boldsymbol{j} + \dfrac{\partial \phi}{\partial z}\boldsymbol{k} = \dfrac{\partial(gz)}{\partial z}\boldsymbol{k} = g\boldsymbol{k}$.

Take $\cdot \boldsymbol{dl}$ on both sides of (2.8a),

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$$\frac{D_a V_a}{Dt} \cdot dl = -\frac{1}{\rho} \nabla p \cdot dl - \nabla \phi \cdot dl . \qquad\qquad (2.8c)$$

Left hand side can be rewritten as

$$\frac{D_a V_a}{Dt} \cdot dl = \frac{D}{Dt}(V_a \cdot dl) - V_a \cdot \frac{D_a}{Dt}(dl)$$

or after observing that since l is a position vector,

$$\frac{D_a l}{Dt} = V_a ,$$

$$\frac{D_a V_a}{Dt} \cdot dl = \frac{D}{Dt}(V_a \cdot dl) - V_a \cdot dV_a \qquad\qquad (4.2)$$

Substituting (4.2) into (2.8)'' leads to

$$\frac{D_a}{Dt}(V_a \cdot dl) - V_a \cdot dV_a = -\frac{1}{\rho} \nabla p \cdot dl - \nabla \varphi \cdot dl$$

Taking a close line integral of the above equation gives

$$\oint \frac{D_a}{Dt}(V_a \cdot dl) - \oint V_a \cdot dV_a = -\oint \frac{\nabla p \cdot dl}{\rho} - \oint \nabla \phi \cdot dl$$
$$\qquad\quad (1) \qquad\qquad (2) \qquad\qquad (3) \qquad\quad (4)$$

Term (1): $\oint \frac{D_a}{Dt}(V_a \cdot dl) = \frac{D_a}{Dt} \oint (V_a \cdot dl) = \frac{D_a C_a}{Dt} = \frac{DC_a}{Dt}$

Term (2): $-\oint V_a \cdot dV_a = -\frac{1}{2} \oint d(V_a \cdot V_a) = 0$

(because close line integral of an exact differential is 0.)

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Term (3): $\quad -\oint \dfrac{\nabla p \cdot d\boldsymbol{l}}{\rho} = -\oint \dfrac{dp}{\rho}$

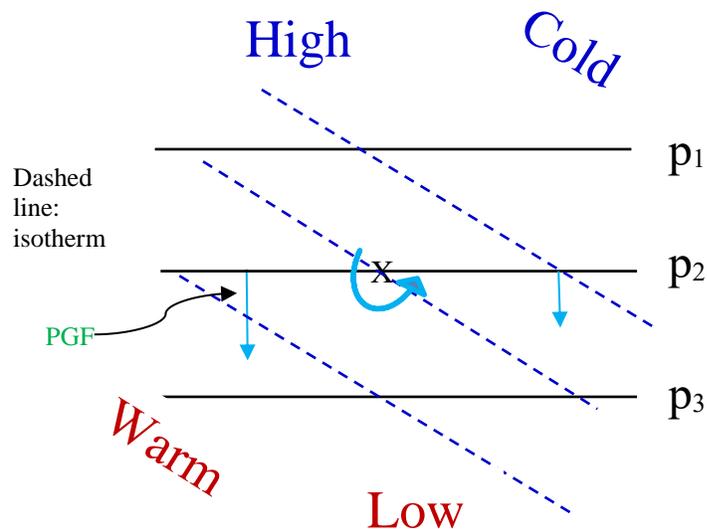Term (4): $\quad -\oint \nabla \phi \cdot d\boldsymbol{l} = -\oint d\phi = 0$.

Thus, we obtain the <u>circulation theorem</u>:

$$\frac{DC_a}{Dt} = -\oint \frac{dp}{\rho} \qquad\qquad (4.3)$$

The term on the right hand side is called "<u>solenoidal term</u>". The physical meaning of the solenoidal term will be explained later.

- Physical meaning of the solenoidal term

$$\frac{DC_a}{Dt} = -\oint \frac{dp}{\rho} \qquad\qquad (4.3)$$

High   Cold

Dashed
line:
isotherm

X

PGF

Warm

Low

p_1

p_2

p_3

2.2 Kelvin's Circulation Theorem

For a barotropic fluid, $\rho = \rho(p,T) = \rho(p)$, there is no temperature difference on an isobaric (pressure) surface. This lead to

$$\frac{DC_a}{Dt} = -\oint \frac{dp}{\rho} = 0.$$

e.g., suppose $\rho = \rho(p) = ap$, then

$$\oint \frac{dp}{\rho} = \oint \frac{dp}{ap} = \frac{1}{a}\oint \frac{dp}{p} = \frac{1}{a}\oint d(\ln p) = \frac{1}{a}\ln p \Big|_{p_o}^{p_o} = 0.$$

(Note that the closed line integral of an exact differential is 0.)

In other words, in a barotropic atmosphere or fluid, the absolute circulation is conserved following the motion, i.e.

$$\frac{DC_a}{Dt} = 0.$$

This is called the Kelvin's circulation theorem.

It can be shown (in homework problem) that Kelvin's circulation theorem is analogous to the conservation of angular momentum.

Recall that
    Linear momentum: $P_{linear} = mv$
    Angular momentum: $L = I\Omega$ where I is the moment of
    Inertia, which depends on the shape of rotating subjects,
and Ω is the angular velocity.

2.3 Bjerknes' Circulation Theorem

For meteorological applications, it is more convenient to use the relative circulation.

Bjerknes' extends Kelvin circulation theorem to the "Bjerknes' circulation theorem".

Recall

$$\boldsymbol{V}_a = \boldsymbol{V} + \boldsymbol{\Omega} \times \boldsymbol{r} \qquad\qquad (2.5)\ (\text{Holton})$$

Taking $\cdot d\boldsymbol{l}$ and integrate along a closed contour on earth's surface gives

$$\underbrace{\oint \boldsymbol{V}_a \cdot d\boldsymbol{l}}_{\substack{\text{Absolute} \\ \text{Circulation}}} = \underbrace{\oint \boldsymbol{V} \cdot d\boldsymbol{l}}_{\substack{\text{Relative} \\ \text{Circulation}}} + \oint (\boldsymbol{\Omega} \times \boldsymbol{r}) \cdot d\boldsymbol{l}$$
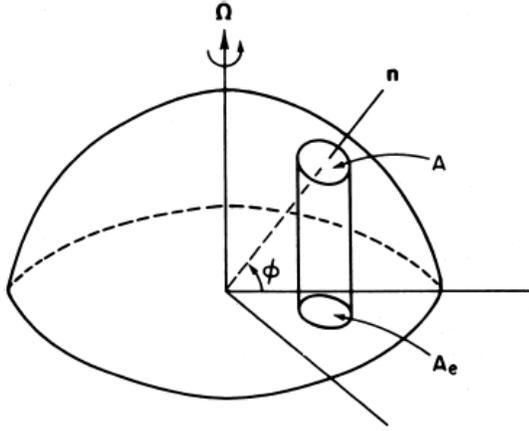
After some manipulation of the second term on the right-hand side, the above equation can be rewritten as

$$C_a = C + 2\Omega A \sin \overline{\phi} = C + 2\Omega A_e,$$

Here A_e defined as

$$A_e = A \sin \overline{\phi}.$$

is the projection of A on equatorial plane as shown below:

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Taking integration of the above equation involving C_a yields

$$\frac{DC_a}{Dt} = \frac{DC}{Dt} + 2\Omega\frac{D(A\sin\bar{\phi})}{Dt}, \text{ or}$$

$$\frac{DC}{Dt} = \frac{DC_a}{Dt} - 2\Omega\frac{D(A\sin\bar{\phi})}{Dt}.$$

Inserting the circulation theorem into the above equation gives the Bjerknes' circulation theorem:

$$\frac{DC}{Dt} = -\oint\frac{dp}{\rho} - 2\Omega\frac{D}{Dt}(A\sin\bar{\phi}) \qquad (4.5)$$

For a barotropic atmosphere (no temperature variation on an isobaric surface), Eq. (4.5) reduces to

$$\frac{DC}{Dt} = -2\Omega\frac{D}{Dt}(A\sin\bar{\phi}).$$

Integrating the above equation from time 1 to 2 leads to

$$\int_1^2 \frac{DC}{Dt}\,dt = -2\Omega \int_1^2 \frac{D}{Dt}(A \sin \bar{\phi})\,dt \, ,$$

or

$$C_2 - C_1 = -2\Omega(A_2 \sin \bar{\phi}_2 - A_1 \sin \bar{\phi}_1). \qquad (4.6)$$

That is, circulation changes if the area of the fluid chain or the latitude changes.

- Applications of Bjerknes circulation theorem.

  Example 1: Consider an area, say A, originally located at equator with no circulation, which moves to the North Pole without changing its area. Estimate the final circulation C. Why does the final circulation become negative?

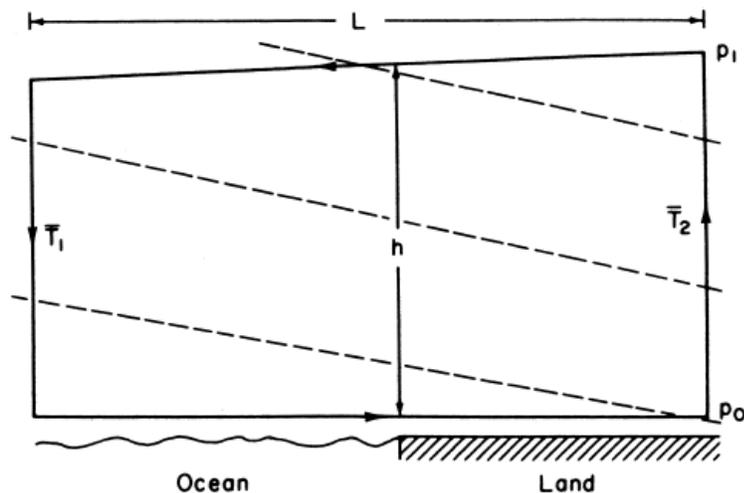 Example 2: Sea-breeze circulation

Fig. 4.3   Application of the circulation theorem to the sea breeze problem. The closed heavy solid
line is the loop about which the circulation is to be evaluated. Dashed lines indicate surfaces
of constant density.

Note that the isothermal surfaces are tilted in opposite way of
the density surface.

$$\frac{DC_a}{Dt} = -\oint \frac{dp}{\rho} = -\oint RT \frac{dp}{p} = -\oint RT \, d(\ln p)$$

$$= -\int_a^b RT \, d(\ln p) - \int_b^c RT \, d(\ln p) - \int_c^d RT \, d(\ln p) - \int_d^a RT \, d(\ln p)$$

Where a denotes the lower left corner, b the lower right corner, c upper right
corner, and d the upper left corner.

Assuming the isobaric (pressure) surface is nearly horizontal,
then the 1st and 3rd terms are approximately 0.

$$\frac{DC_a}{Dt} = -\int_b^c RT \, d(\ln p) - \int_d^a RT \, d(\ln p) = -R\bar{T}_2 \int_b^c d(\ln p) - R\bar{T}_1 \int_d^a d(\ln p)$$

$$\frac{DC_a}{Dt} = -R\bar{T}_2 \ln \frac{p_c}{p_b} - R\bar{T}_1 \ln \frac{p_a}{p_d} = R(\bar{T}_2 - \bar{T}_1) \ln \frac{p_o}{p_1} \, .$$