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Sec. 2.2 Scale Analysis and Approximations of the Continuity Equation

The governing equations of atmospheric motion and processes are based on the following physical laws (*Ch. 2, Lin 2007*):

(a) Newton's second law of motion,
(b) Conservation of mass, and
(c) First law of thermodynamics.

These laws are represented by the set of primitive equations that are comprised by

(a) Horizontal and vertical momentum equations
(b) Continuity equation, and
(c) Thermodynamic energy equation.

Note that wave motions behave completely differently from mass transport.

Briefly speaking, fluid particles do not necessarily follow the disturbance in wave motion, while they do always follow it in mass transport. For example, air parcels associated with gravity waves may oscillate in the vicinity of the source or forcing region, but the gravity waves themselves may propagate to great distances from their origin.

On the other hand, the air parcels within a cold pool generated by evaporative cooling - associated with falling raindrops beneath a thunderstorm - always move in concert with the density current.

Considering an atmosphere on a planetary *f* plane, the momentum equations, continuity equation, and thermodynamic energy equation can be expressed in the following form (Lin 2007):

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_{rx}, \tag{2.2.1}$$

$$\frac{Dv}{Dt} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + F_{ry}, \tag{2.2.2}$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + F_{rz} \tag{2.2.3}$$

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0, \tag{2.2.4}$$

$$\frac{D\theta}{Dt} = \frac{\theta}{c_p T}\dot{q}, \tag{2.2.5}$$

where

$D/Dt = \partial/\partial t + u\partial/\partial x + v\partial/\partial y + w\partial/\partial z$: (total or material derivative) the change of a certain property within a fluid parcel following the motion.

F_{rx}, F_{ry}, and F_{rz} : viscous terms or frictional forces per unit mass in the *x*, *y*, and *z* directions, respectively.

c_p : heat capacity of dry air at constant pressure (1004 J/kg-K), and

$\dot{q}$: diabatic heating rate in $J\,kg^{-1}s^{-1}$.

Other symbols are defined as usual (e.g., see Appendix A of Lin 2007).

In the viscous sublayer, which is a very thin layer of O(cm) near the earth's surface, the viscous terms may be represented by molecular viscosity in the form $\nu\nabla^2 u, \nu\nabla^2 v,$ and $\nu\nabla^2 w$, kinematic viscosity coefficient associated with molecular viscosity. Note that ν is equal to μ/ρ, where

μ is the dynamic viscosity coefficient and ρ is the air density. At sea level, ν has a value of about $1.46x10^{-6}\ m^2 s^{-1}$.

The molecular viscosity is almost completely negligible in the atmosphere above the viscous sublayer, where momentum and heat transfers are dominated by turbulent eddy motion. A number of parameterization schemes for turbulent eddy viscosity in the planetary boundary layer will be discussed in Ch. 5 of Holton (2005).

The equation set (2.2.1-2.2.3) with no Coriolis terms is often referred to as the Navier-Stokes equations of motion. The diabatic heating rate may be taken to represent, for example, surface sensible heating, elevated latent heating, or cloud-top radiative cooling.

Note that the viscous terms on the right-hand side of Eqs. (2.2.1) and (2.2.2) can be approximated by Rayleigh friction ($-\nu_o u$, $-\nu_o v$), while the diabatic heating term of Eq. (2.2.5) can be approximated by the Newtonian cooling ($-\nu_o \theta$), as is done in some theoretical studies to simplify the above system of governing equations. The coefficient ν_o is determined by the e-folding time scale of the disturbance.

In the above system, Eqs. (2.2.1) - (2.2.5), there are seven unknowns represented by five equations. In order to close the system, we need two additional equations. Two equations can be used to close the system:

(1) The equation of state for dry air (which is well represented by an ideal gas law),

$$p = \rho R_d T \tag{2.2.6}$$

(2) The Poisson's equation

$$\theta = T(p_o / p)^{R_d / c_p}, \tag{2.2.7}$$

where

θ : potential temperature (the temperature a dry air parcel would be when it is taken to 1000 mb)

p_o : a constant reference pressure level (normally chosen as 1000 mb) and

R_d : the gas constant for dry air (287 J/kg-K).

For a moist atmosphere, the temperature in Eq. (2.2.6) is replaced by the virtual temperature, which takes into account the moist effects due to latent heat release, and the density is replaced by the total density, which is a sum of the dry air density and the total water density.

To formulate a more complete atmospheric system, we need to include nonlinear advective accelerations, viscosity and conservation equations for water substances (e.g. water vapor, cloud water, rain, ice, snow, and hail) in addition to the system of Eqs. (2.2.1) – (2.2.5).

- Derivation of Continuity Equation

2.5.1 A Eulerian Approach

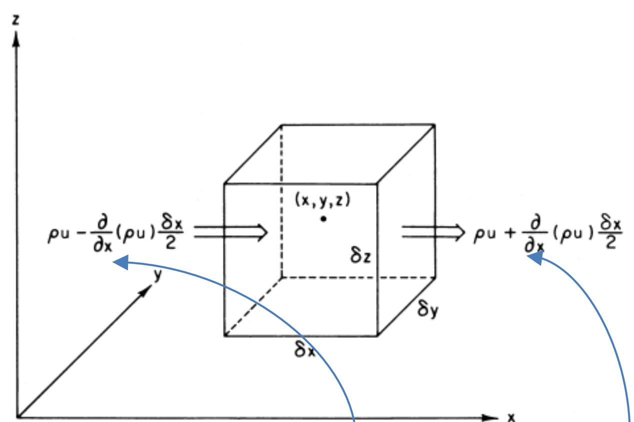

Fig. 2.5 Mass inflow into a fixed (Eulerian) control volume due to motion parallel to the x axis.

A simple alternative for this Eulerian approach is to

(a) move the (x, y, z) to the center of the left side of the control volume, and

(b) replace the mass inflow and outflow in Fig. 2.5 by ρu and $\rho u + \frac{\partial}{\partial x}(\rho u)\delta x$, respectively.

We consider a volume element $\delta x\ \delta y\ \delta z$ that is fixed in a Cartesian coordinate frame as shown in Fig. 2.5. For such a fixed control volume the net rate of mass inflow through the sides must equal the rate of accumulation of mass within the volume. The rate of inflow of mass through the left-hand face per unit area is

$$\left[\rho u - \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]$$

whereas the rate of outflow per unit area through the right-hand face is

$$\left[\rho u + \frac{\partial}{\partial x}(\rho u)\frac{\delta x}{2}\right]$$

ρu $\qquad\qquad$ $\rho u + \frac{\partial}{\partial x}(\rho u)\delta x$

Because the area of each of these faces is $\delta y \delta z$, the net rate of mass flow into the volume due to the x velocity component is

$$(\rho u)\delta y\delta z - \left[\rho u + \frac{\partial}{\partial x}(\rho u)\right]\delta y\delta z = -\frac{\partial}{\partial x}(\rho u)\delta x\delta y\delta z$$

Similar expressions obviously hold for the y and z directions. Thus, the net rate of mass inflow is

$$-\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]\delta x\delta y\delta z$$

Thus, the local rate of change of the mass of the control volume is

$$\frac{\partial m}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]\delta x\delta y\delta z$$

Dividing the above equation by $\delta x\delta y\delta z$ (=δV) leads to

$$\frac{\partial \rho}{\partial t} = -\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]$$

Or

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\rho\boldsymbol{U} = 0 \qquad (2.30)$$

Eq. (2.30) is the mass divergence form of the continuity equation.

An alternative form of the continuity equation is obtained by applying the vector identity

$$\nabla \cdot (\rho \mathbf{U}) \equiv \rho \nabla \cdot \mathbf{U} + \mathbf{U} \cdot \nabla \rho$$

and the relationship

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla$$

to get

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \nabla \cdot \mathbf{U} = 0 \tag{2.31}$$

Equation (2.31) is the velocity divergence form of the continuity equation. It states that the fractional rate of increase of the density *following the motion* of an air parcel is equal to minus the velocity divergence. This should be clearly distinguished from (2.30), which states that the *local* rate of change of density is equal to minus the mass divergence.

2.5.2 A Lagrangian Derivation

The physical meaning of divergence can be illustrated by the following alternative derivation of (2.31). Consider a control volume of fixed mass δM that moves with the fluid. Letting $\delta V = \delta x\ \delta y \delta z$ be the volume, we find that because $\delta M = \rho \delta V = \rho \delta x \delta y \delta z$ is conserved following the motion, we can write

$$\frac{1}{\delta M}\frac{D}{Dt}(\delta M) = \frac{1}{\rho \delta V}\frac{D}{Dt}(\rho \delta V) = \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{\delta V}\frac{D}{Dt}(\delta V) = 0 \tag{2.32}$$

but

$$\frac{1}{\delta V}\frac{D}{Dt}(\delta V) = \frac{1}{\delta x}\frac{D}{Dt}(\delta x) + \frac{1}{\delta y}\frac{D}{Dt}(\delta y) + \frac{1}{\delta z}\frac{D}{Dt}(\delta z)$$

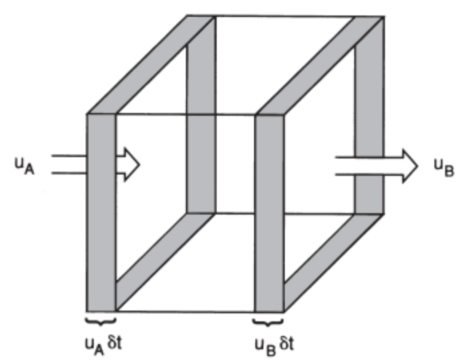

Fig. 2.6 Change in Lagrangian control volume (shown by shading) due to fluid motion parallel to the x axis.

so that in the limit $\delta V \to 0$, (2.32) reduces to the continuity equation (2.31); the divergence of the three-dimensional velocity field is equal to the fractional rate of change of volume of a fluid parcel in the limit $\delta V \to 0$. It is left as a problem for the student to show that the divergence of the *horizontal* velocity field is equal to the fractional rate of change of the horizontal area δA of a fluid parcel in the limit $\delta A \to 0$.

- ***Scale Analysis of the Continuity Equation***

Letting $\rho(t,x,y,z)=\bar{\rho}(z)+\rho'(t,x,y,z)$, the continuity equation (2.2.4) reduces to the perturbation form

$$\frac{\partial \rho'}{\partial t}+V\cdot\nabla\rho'+w\frac{d\bar{\rho}}{dz}+(\bar{\rho}+\rho')\nabla\cdot V=0, \qquad (2.2.20)$$

where $\boldsymbol{V}=(u,v,w)$, or

$$\frac{1}{\bar{\rho}}\frac{\partial\rho'}{\partial t} + \frac{1}{\bar{\rho}}\boldsymbol{V}\cdot\nabla\rho' + \frac{w}{\bar{\rho}}\frac{d\bar{\rho}}{dz} + \nabla\cdot\boldsymbol{V} + \frac{\rho'}{\bar{\rho}}\nabla\cdot\boldsymbol{V}=0 \qquad (2.2.21)$$

Scales

$$\frac{\rho'}{\rho_o}\frac{U}{L} \qquad \frac{\rho'}{\rho_o}\frac{U}{L} \qquad \frac{W}{H} \qquad \frac{U}{L} \qquad \frac{\rho'}{\rho_o}\frac{U}{L} \qquad (2.2.22)$$

$$(\text{or } \frac{\rho'}{\rho_o}\frac{W}{L_z}) \qquad\qquad (\text{or } \frac{W}{L_z}) \quad (\text{or } \frac{\rho'}{\rho_o}\frac{W}{L_z})$$

Characteristic magnitudes (m/s^2) for midlatitude synoptic motion:

$U\sim 10$ m/s, $L\sim 1000$ km (10^6 m), $\rho_o\sim 1$ kg/m^3, $\rho'\sim .01$ kg/m^3,

$H\sim 10$ km, $L_z\sim 10$ km (deep convection) or 1 km (shallow convection),

W ~ 0.01 m/s

It is important to distinguish the difference between the scale height (H) and the vertical scale of convection or disturbance (L_z); the former (H) is controlled by the basic structure of the atmosphere while the latter (L_z) is controlled by the fluid motion.

Anelastic and incompressible approximations to the continuity equation can also be obtained by applying scale analysis to (2.2.17). Unlike that used in Holton (2004, Ed. 4), here we make a difference in H and L_z, as mentioned above.

For shallow convection or disturbance

10^{-7} 10^{-7} (or 10^{-7}) 10^{-6} 10^{-5} (or10^{-5}) 10^{-7}(or 10^{-7}) (2.2.23)

For deep convection or disturbance

10^{-7} 10^{-7} (or 10^{-8}) 10^{-6} 10^{-5} (or10^{-6}) 10^{-7}(or 10^{-8}) (2.2.24)

➢ *Anelastic approximation*

Keeping the terms of O(10^{-5}) & O(10^{-6}) leads to the anelastic (deep) convection continuity equation:

$$\frac{w}{\bar{\rho}}\frac{d\bar{\rho}}{dz} + \nabla\cdot\boldsymbol{V} = 0. \qquad (2.3.1)'$$

The effect of this approximation is to eliminate all waves with very high propagation speeds associated with rapid (adiabatic) compression and expansion of the fluid.

The above anelastic (deep-convection) continuity equation may also be written as

$$\nabla\cdot\boldsymbol{V'} - \frac{w'}{H} = 0, \qquad (2.3.1)$$

or

$$\nabla\cdot(\boldsymbol{V'}e^{-z/H}) = 0, \qquad (2.3.2)$$

since the scale height is taken to be a constant. Equations (2.3.1) and (2.3.2) may also be expressed in an alternative form:

$$\nabla\cdot(\bar{\rho}\boldsymbol{V'}) = 0. \qquad (2.3.3)$$

Note that (2.3.3) is linked with (2.3.2) when the density decays exponentially with height, with an e-folding value of scale height *H*.

Equations (2.3.1), (2.3.2), or (2.3.3) are called the anelastic or deep convection continuity equations.

➤ *Incompressible approximation*

If we keep only the largest terms ($O(10^{-5})$), it leads to the incompressible (shallow convection) continuity equation:

$$\nabla \cdot \boldsymbol{V} = 0 . \tag{2.3.4}$$

This means that conservation of mass has become conservation of volume because density is treated as a constant. Thus, volume is a good proxy for mass under this approximation.

[Reading Assignment] (From Ch. 2 of Lin 2007)

"Equation (2.3.3) was first proposed by Batchelor (1953), who defined $\bar{\rho}(z)$ to be the density in an adiabatic, stably stratified, horizontally uniform reference state. The name anelastic was coined by Ogura and Phillips (1962), who derived (2.3.3) through a rigorous scale analysis, along with approximate forms for the momentum and thermodynamic energy equations.

Their scaling analysis assumes that: (a) all deviations of the potential temperature θ' from some constant mean value θ_o are small, and (b) the time scale of the disturbance is comparable to the time scale for gravity wave oscillations. The terms that are neglected in the original anelastic equations are an order $\varepsilon = \theta'/\theta_o$ smaller than those that are retained. Thus, in the case of dry convection (where mixing will keep the environmental lapse rate close to the adiabatic lapse rate), ε will be small and the anelastic equations can be used to represent nonacoustic modes with complete confidence. For deep, moist convection or gravity wave propagation, however, the mean-state static stability can be sufficient to make ε rather large. For example, the θ' variations across a 10 km deep isothermal layer may reach as high as 40% of the mean θ_o.

Equation (2.3.1) may be further simplified, by assuming that the vertical scale (L_z) of the mesoscale disturbance is much smaller than the scale height of the basic state atmosphere, $L_z / H << 1$,

$$\nabla \cdot \boldsymbol{V'} = 0 . \tag{2.3.4}$$

The above equation is called the incompressible or shallow convection continuity equation. This means that conservation of mass has become conservation of volume because density is treated as a constant. Thus, volume is a good proxy for mass under this approximation.

Again, it is important to distinguish the difference between the scale height (H) and the vertical scale of convection or of the disturbance (L_z) because the scale height is controlled by the basic structure of the atmosphere, instead of by the fluid motion. Anelastic and incompressible approximations to the continuity equation can also be obtained by applying scale analysis to 2.2.17, similar to that used in Holton (2004), except that it is necessary to differentiate H and L_z, as mentioned above."