ASME 434 Atmospheric Dynamics II   Dr. Yuh-Lang Lin

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Chapter 4 The Vorticity Equation

- Derivation of the vorticity equation

Cross-differentiating the zonal and meridional component equations with respect to x and y gives:

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}\right) \qquad (4.10)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}\right) \qquad (4.11)$$

Subtracting (4.10) from (4.11) and recalling that $\zeta = \partial v/\partial x - \partial u/\partial y$ lead to the vorticity equation:

$$\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}+w\frac{\partial \zeta}{\partial z}+(\zeta + f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+$$

$$\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+v\frac{\partial f}{\partial y}=\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) \qquad (4.12)$$

or

$$\frac{\partial \zeta}{\partial t}=-\boldsymbol{V}\cdot\nabla(\zeta + f)-(\zeta + f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)$$

$$+v\frac{\partial f}{\partial y}+\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y}-\frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

Using $\frac{Df}{Dt}=v\frac{\partial f}{\partial y}$, (4.16) can also be rewritten in the form

$$\frac{D}{Dt}(\zeta + f) = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) \quad (4.13)$$

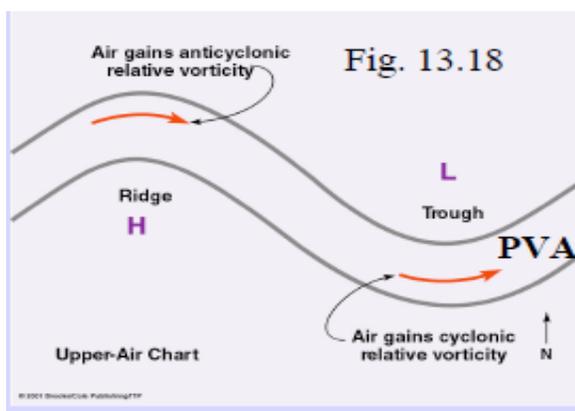- Physical meaning of individual terms of the vorticity equation Eq. (4.17) may also be written as

$$\frac{\partial \zeta}{\partial t} = -\boldsymbol{V}\cdot\nabla(\zeta + f) - (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Local rate of change of vertical vorticity   Vorticity advection   Divergence (stretching) term

$$-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right) + \frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right)$$

Tilting (twisting) term   Solenoidal term
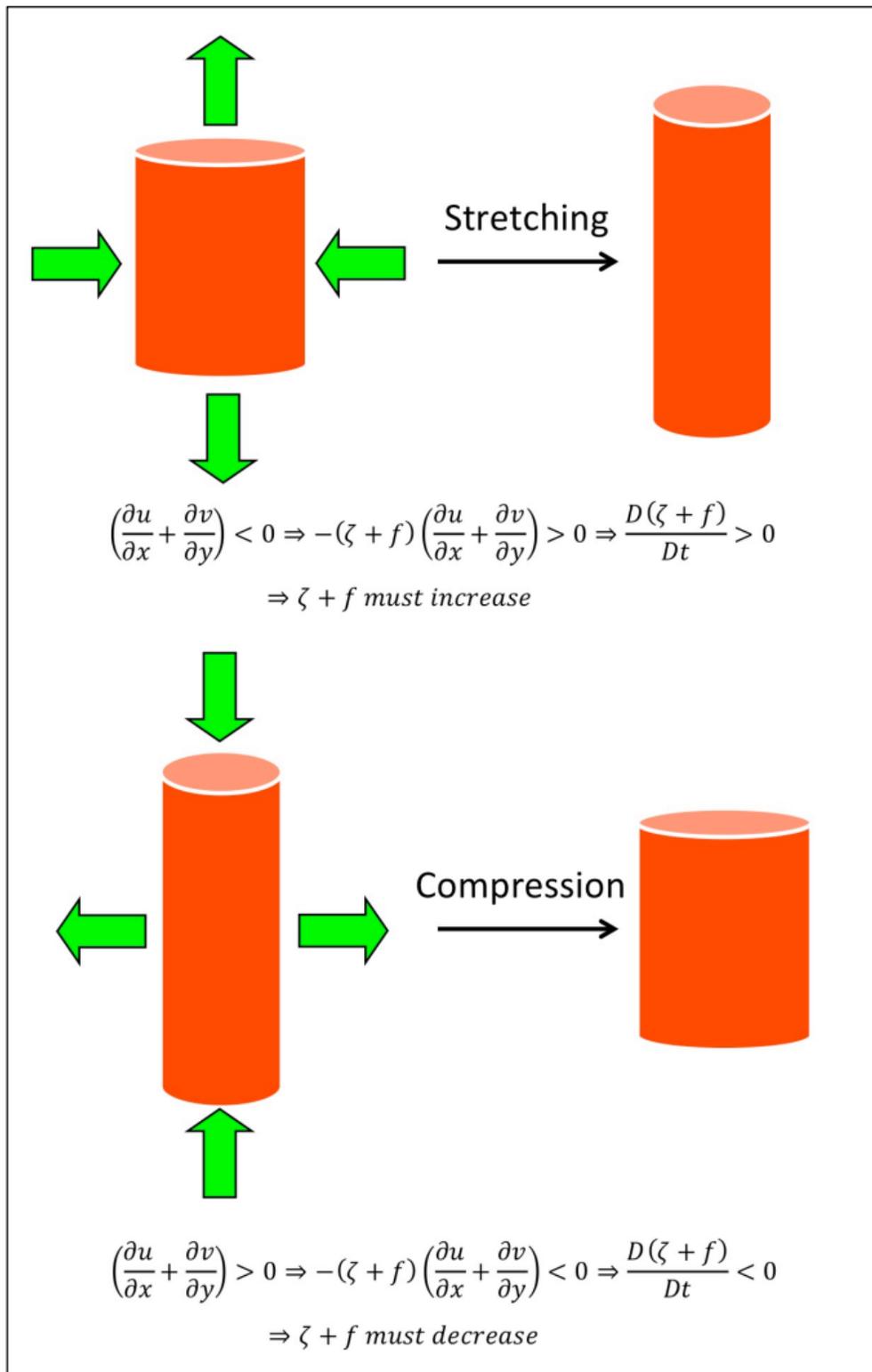
- Physical interpretation of the vorticity advection term

$$-V\cdot\nabla(\zeta + f) = \left(-u\frac{\partial \zeta}{\partial x} - v\frac{\partial \zeta}{\partial y}\right) - \beta v \qquad \text{(where } \beta = \partial \zeta / \partial y)$$
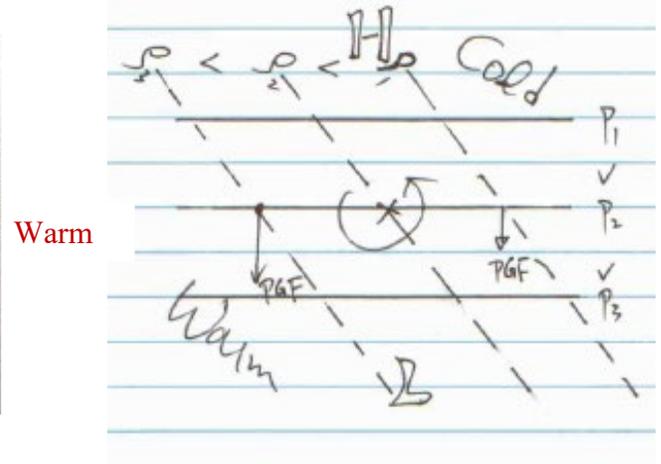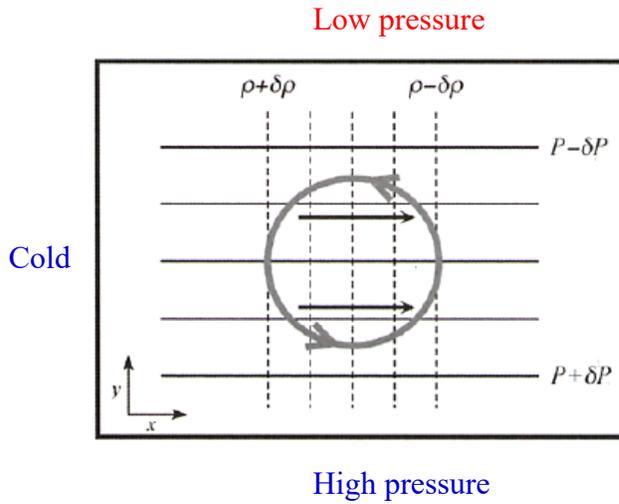

Fig. 13.18

	<0	>0
$\frac{\partial \zeta}{\partial t} \propto -u\frac{\partial \zeta}{\partial x}:$	<0	>0
$\frac{\partial \zeta}{\partial t} \propto -\beta v:$	>0	<0

- Physical interpretation of the divergence (stretching) term

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) < 0 \Rightarrow -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) > 0 \Rightarrow \frac{D(\zeta + f)}{Dt} > 0$$

$$\Rightarrow \zeta + f \text{ must increase}$$

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) > 0 \Rightarrow -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) < 0 \Rightarrow \frac{D(\zeta + f)}{Dt} < 0$$

$$\Rightarrow \zeta + f \text{ must decrease}$$

- Physical interpretation of the solenoidal term (also see the section of sea-breeze circulation)

$$\frac{1}{\rho^2}\left(\frac{\partial \rho}{\partial x}\frac{\partial p}{\partial y} - \frac{\partial \rho}{\partial y}\frac{\partial p}{\partial x}\right) \quad > 0$$

+   -   -

Thus, the solenoidal term will induce increase of ζ.

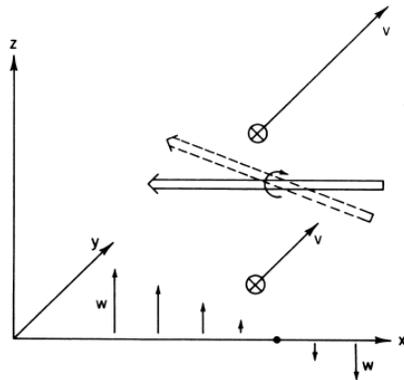- Physical interpretation of the tilting term

Fig. 4.12 Vorticity generation by the tilting of a horizontal vorticity vector (double arrow).

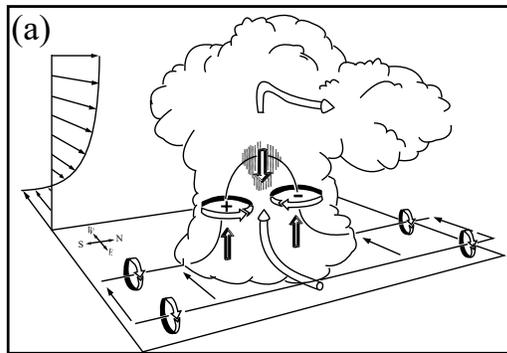## Application to the formation of mesocyclones and tornadoes

Fig. 8.20: A schematic depicting rotation development and the storm splitting. (a) Rotation development: In the early stage, a pair of vortices forms through tilting of horizontal vorticity associated with the (westerly) environmental shear. (b) Storm splitting: In the later stage, the precipitation induced downdraft splits the updraft. At this stage, vortex lines are tilted downward, producing two vortex pairs. Cylindrical arrows denote the direction of the storm-relative airflow, and heavy solid lines represent vortex lines with the sense of rotation denoted by circular arrows. Shaded arrows represent the forcing promoting new updraft and downdraft acceleration. Vertical dashed lines denote regions of precipitation. Frontal symbols at the surface mark the boundary of cold air outflow. (Lin 2007, After Klemp)

- Tornadogenesis Dynamics

  - A mesocyclone developed by tilting horizontal vorticity into the vertical, which provides the rotation of a thunderstorm, called supercell thunderstorm.
  - Within the supercell thunderstorm, extremely strong vorticity may be developed due to vertical stretching associated with latent heating, which may produce tornadoes.

Example shown is for purely helical flow (all the horizontal vorticity is streamwise)
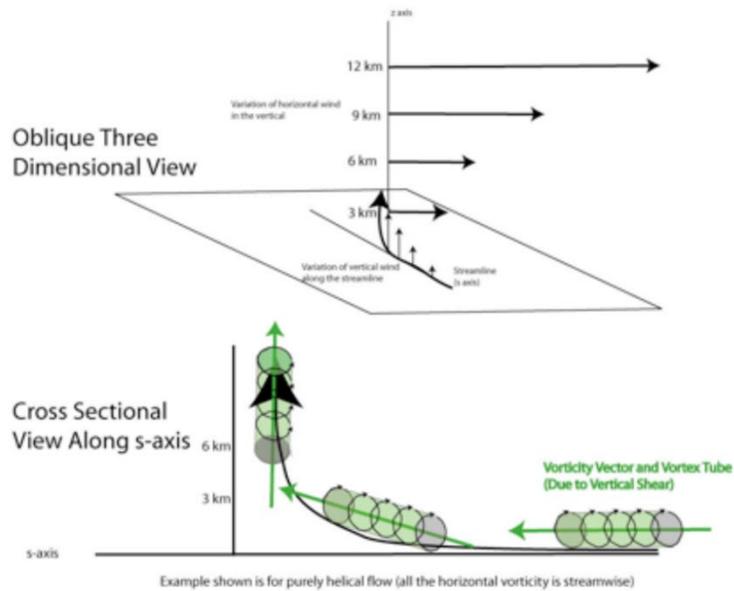
The above process, i.e., phasing the vorticity with velocity can be represented by helicity H:

$$H = \boldsymbol{\omega} \cdot \boldsymbol{V}$$

The above definition can be expanded out to:
Note that 3D helicity is the product of the 3 wind components with

$$H = u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - v\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \quad (1)$$

the three components of vorticity. Note that the far-right hand term is a product of ζ and w, and is generated by tilting the first two terms are generated by vertical wind shear in tornadogenesis.

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- Vorticity Equation in Isobaric (pressure) Coordinates

$$\frac{\partial \zeta}{\partial t} = -V \cdot \nabla (\zeta + f) - \omega \frac{\partial \zeta}{\partial p} - (\zeta + f)\nabla \cdot V + k \cdot \left(\frac{\partial V}{\partial p} \times \nabla \omega \right) \qquad (4.21)$$

Note the solenoidal term is implicit.

- Scale analysis of the Vorticity Equation

Based on typical observed magnitudes for synoptic-scale motions, scales are chosen as follows:

$$U \sim 10\,\text{m s}^{-1} \qquad \text{horizontal scale}$$
$$W \sim 1\,\text{cm s}^{-1} \qquad \text{vertical scale}$$
$$L \sim 10^6\,\text{m} \qquad \text{length scale}$$
$$H \sim 10^4\,\text{m} \qquad \text{depth scale}$$
$$\delta p \sim 10\,\text{hPa} \qquad \text{horizontal pressure scale}$$
$$\rho \sim 1\,\text{kg m}^{-3} \qquad \text{mean density}$$
$$\delta \rho / \rho \sim 10^{-2} \qquad \text{fractional density fluctuation}$$
$$L/U \sim 10^5\,\text{s} \qquad \text{time scale}$$
$$f_0 \sim 10^{-4}\,\text{s}^{-1} \qquad \text{Coriolis parameter}$$
$$\beta \sim 10^{-11}\,\text{m}^{-1}\,\text{s}^{-1} \qquad \text{“beta” parameter}$$

The vorticity equation may be approximated by

$$\frac{D_h (\zeta + f)}{Dt} = -(\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \qquad (4.22b)$$

Where $D_h / Dt \equiv \partial / \partial t + u \partial / \partial x + v \partial / \partial y$.