

8.(3.1) Elementary Applications of the Basic Equations

8.1 Basic Equations in Isobaric Coordinates

(Ref.: Holton Sec. 3.1)

- **The Horizontal Momentum Equation**

The approximate horizontal momentum equations (2.24) and (2.25)

$$\frac{Du}{Dt} = fv - \frac{1}{\rho} \frac{\partial p}{\partial x} \quad (2.24)$$

$$\frac{Dv}{Dt} = -fu - \frac{1}{\rho} \frac{\partial p}{\partial y} \quad (2.25)$$

Inertial Force
Coriolis Force
PGF

may be written in the isobaric (pressure) coordinates by replacing the following PGF

$$-\frac{1}{\rho} \left(\frac{\partial p}{\partial x} \right)_z = - \left(\frac{\partial \phi}{\partial x} \right)_p ; \quad -\frac{1}{\rho} \left(\frac{\partial p}{\partial y} \right)_z = - \left(\frac{\partial \phi}{\partial y} \right)_p \quad (1.20)$$

into them

$$\frac{Du}{Dt} = fv - \frac{\partial \phi}{\partial x}, \quad \frac{Dv}{Dt} = -fu - \frac{\partial \phi}{\partial y} \quad (3.2a-b)$$

Equations (2.24)-(2.25) and (3.2a)-(3.2b) can be written in vectorial form as

$$\frac{DV}{Dt} + f \mathbf{k} \times \mathbf{V} = -\frac{1}{\rho} \nabla_p p \quad (3.1)$$

$$\frac{DV}{Dt} + f \mathbf{k} \times \mathbf{V} = -\nabla_p \phi \quad (3.2)$$

where $\mathbf{V} = u \mathbf{i} + v \mathbf{j}$ is the horizontal velocity vector and ∇_p is the horizontal gradient operator applied with pressure held constant.

Because p is the independent vertical coordinate, we must expand the total derivative as

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \frac{Dx}{Dt} \frac{\partial}{\partial x} + \frac{Dy}{Dt} \frac{\partial}{\partial y} + \frac{Dp}{Dt} \frac{\partial}{\partial p} = \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + \omega \frac{\partial}{\partial p} \quad (3.3)$$

Here $\omega = Dp / Dt$ is the “[omega vertical motion](#),” which is defined as the pressure change following the motion, similar to $w = Dz / Dt$. [For synoptic motions](#), $\omega \approx -\rho g w$.

- From (3.2), the geostrophic relationship can be written as

$$f \mathbf{V}_g = \mathbf{k} \times \nabla_p \phi \quad (3.4)$$

or in scalar form

$$f u_g = -\frac{\partial \phi}{\partial y}, \quad f v_g = \frac{\partial \phi}{\partial x}. \quad (3.4a-b)$$

Note there is no density present in (3.4). In addition, on an f -plane (i.e., f is constant), we have

$$\nabla_p \cdot \mathbf{V}_g = 0$$

That is, there is no divergence for the geostrophic flow (non-divergent).

- The continuity equation in the isobaric coordinates becomes

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)_p + \frac{\partial \omega}{\partial p} = 0 \quad (3.5)$$

- The thermodynamic energy equation in the isobaric coordinates becomes

$$c_p \frac{DT}{Dt} - \alpha \frac{Dp}{Dt} = J \quad (2.42)$$

then becomes

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} - S_p \omega = \frac{J}{c_p} \quad (3.6)$$

where $J = \frac{Dq}{Dt}$ is the diabatic heating rate and

$$S_p \equiv \frac{RT}{c_p p} - \frac{\partial T}{\partial p} = -\frac{T}{\theta} \frac{\partial \theta}{\partial p}, \text{ or } S_p \equiv \frac{\Gamma_d - \Gamma}{\rho g} \quad (3.7)$$

where S_p is the “static stability parameter”.