

3.(1.3) Apparent Forces (Equation of Motion in Rotating Frame of Reference)

Objective: To apply Newton's 2nd Law in a non-inertial frame of reference associated with Earth's rotation.

In order to do so, two additional apparent or virtual forces are emerged from the derivations: [centrifugal force](#) and [Coriolis force](#).

(a) Total differentiation of a scalar

Consider releasing a balloon at a place with $T = T_o(x_o, y_o, z_o, t_o)$ to another place with $T + \delta T = T(x_o + \Delta x, y_o + \Delta y, z_o + \Delta z, t_o + \Delta t)$, the temperature change following the balloon may be derived by applying the [chain rule](#),

$$\Delta T = \left(\frac{\partial T}{\partial t}\right) \Delta t + \left(\frac{\partial T}{\partial x}\right) \Delta x + \left(\frac{\partial T}{\partial y}\right) \Delta y + \left(\frac{\partial T}{\partial z}\right) \Delta z.$$

Dividing the above equation by Δt and taking $\lim_{\delta T \rightarrow 0}$ lead to

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u \left(\frac{\partial T}{\partial x}\right) + v \left(\frac{\partial T}{\partial y}\right) + w \left(\frac{\partial T}{\partial z}\right) \quad \text{or} \quad \frac{\partial T}{\partial t} = \frac{DT}{Dt} - \mathbf{V} \cdot \nabla T. \quad (2.5)$$

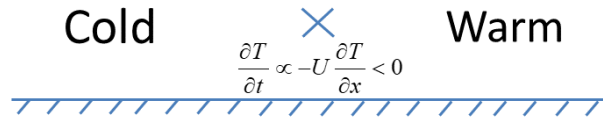
[Physical meaning](#):

$\frac{DT}{Dt}$: Total rate of change of temperature following the motion

$\frac{\partial T}{\partial t}$: Local rate of change of temperature at a fixed location

$-V \cdot \nabla T$: Temperature advection

$$\frac{\partial T}{\partial t} = \frac{DT}{Dt} - U \frac{\partial T}{\partial x}$$

$$\begin{array}{ccc} \text{Cold} & \times & \text{Warm} \\ & \frac{\partial T}{\partial t} \propto -U \frac{\partial T}{\partial x} < 0 & \end{array}$$


Example: Cold advection associated with a flow from cold to warm region.

(b) Total differentiation of a vector in a rotating system

For any vector $\mathbf{A}$, decompose it into three components in an inertial frame of reference or a rotating frame of reference (e.g., on a merry-go-around),

$$\mathbf{A} = A_x \mathbf{i} + A_y \mathbf{j} + A_z \mathbf{k} = A'_x \mathbf{i}' + A'_y \mathbf{j}' + A'_z \mathbf{k}'$$

(inertial frame) (rotating frame)

Taking the total derivative of $\mathbf{A}$ gives

$$\begin{aligned} \frac{D_a \mathbf{A}}{Dt} &= \frac{DA_x}{Dt} \mathbf{i} + \frac{DA_y}{Dt} \mathbf{j} + \frac{DA_z}{Dt} \mathbf{k} \quad \left(\text{Note } \frac{D_a A_x}{Dt} = \frac{DA_x}{Dt} \text{ for scalars} \right) \\ &= \frac{DA'_x}{Dt} \mathbf{i}' + \frac{DA'_y}{Dt} \mathbf{j}' + \frac{DA'_z}{Dt} \mathbf{k}' + A'_x \frac{D\mathbf{i}'}{Dt} + A'_y \frac{D\mathbf{j}'}{Dt} + A'_z \frac{D\mathbf{k}'}{Dt} \\ &= \frac{D\mathbf{A}}{Dt} + \left(A'_x \frac{D\mathbf{i}'}{Dt} + A'_y \frac{D\mathbf{j}'}{Dt} + A'_z \frac{D\mathbf{k}'}{Dt} \right) \end{aligned}$$

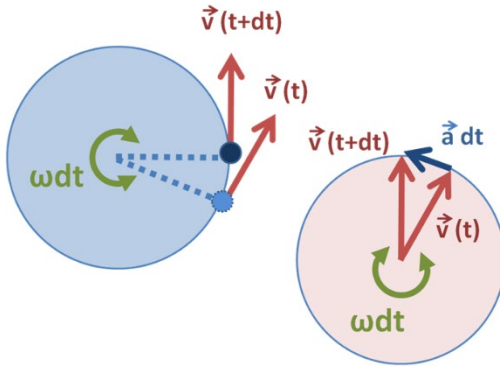


Total derivative of $\mathbf{A}$ in the rotating frame

The above equation leads to

$$\frac{D_a \mathbf{A}}{Dt} = \frac{D\mathbf{A}}{Dt} + \boldsymbol{\Omega} \times \mathbf{A}, \quad (2.6)$$

where $\boldsymbol{\Omega}$ is the rotation vector. (Example: Think about [centripetal force](#) in a circular motion. See below, but let $\boldsymbol{\Omega}$ be ω .)



Applying (2.6) to a position vector $\mathbf{r}$ leads to

$$\frac{D_a \mathbf{r}}{Dt} = \frac{D\mathbf{r}}{Dt} + \boldsymbol{\Omega} \times \mathbf{r} \quad \text{or} \quad \mathbf{V}_a = \mathbf{V} + \boldsymbol{\Omega} \times \mathbf{r}. \quad (2.7)$$

Applying (2.6) to (2.7) again leads to

$$\frac{D_a \mathbf{V}_a}{Dt} = \frac{D\mathbf{V}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{V} - \Omega^2 \mathbf{R} \quad (2.8)$$

Combining Eqs. (2.4) and (2.8) leads to the equation of motion in the rotating frame of reference,

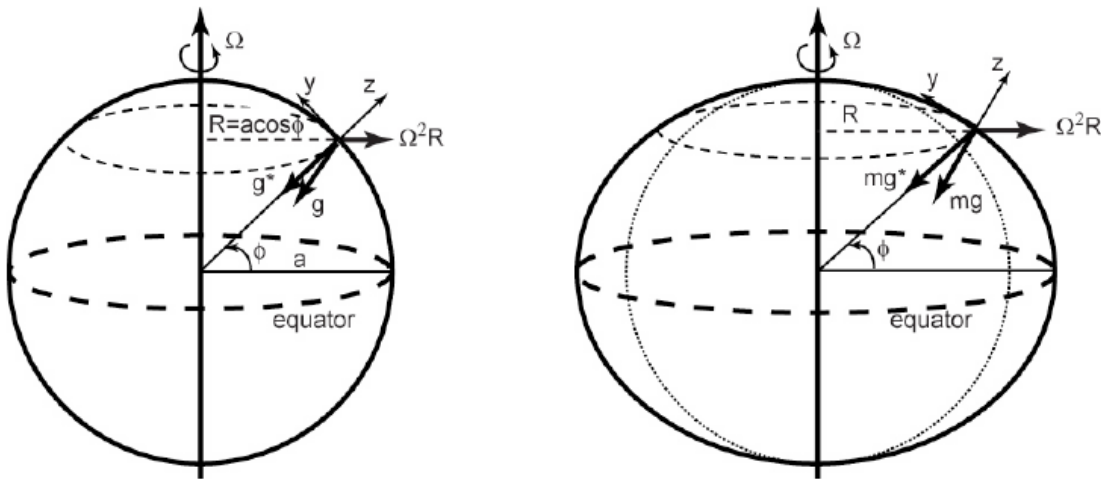
$$\frac{D_a \mathbf{V}_a}{Dt} = \frac{D\mathbf{V}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{V} - \Omega^2 \mathbf{R} = -\frac{1}{\rho} \nabla p + \mathbf{g}^* + \mathbf{F}_r \quad (2.9)$$

Centripetal force (real)

or

$$\frac{D\mathbf{V}}{Dt} = -\frac{1}{\rho}\nabla p - \underbrace{2\boldsymbol{\Omega} \times \mathbf{V}}_{\text{Coriolis force}} + \underbrace{\mathbf{g} + \mathbf{F}_r}_{\text{Centrifugal force (apparent)}}, \quad (2.10)$$

where $\mathbf{g} = \mathbf{g}^* + \Omega^2 \mathbf{R}$ is the effective gravity.



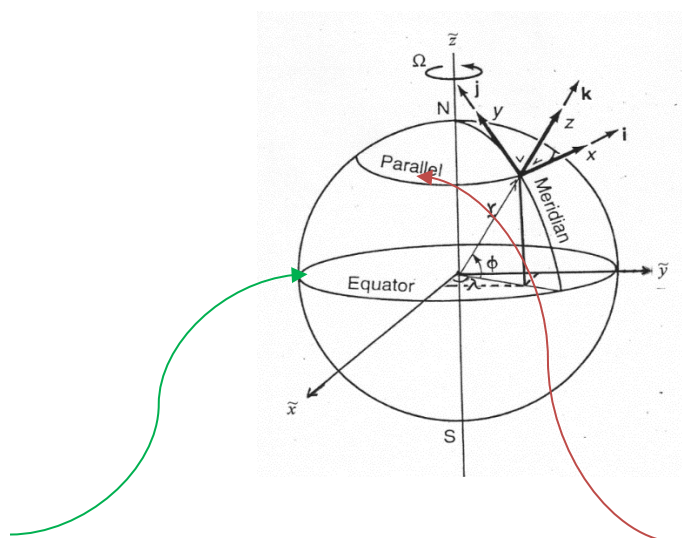
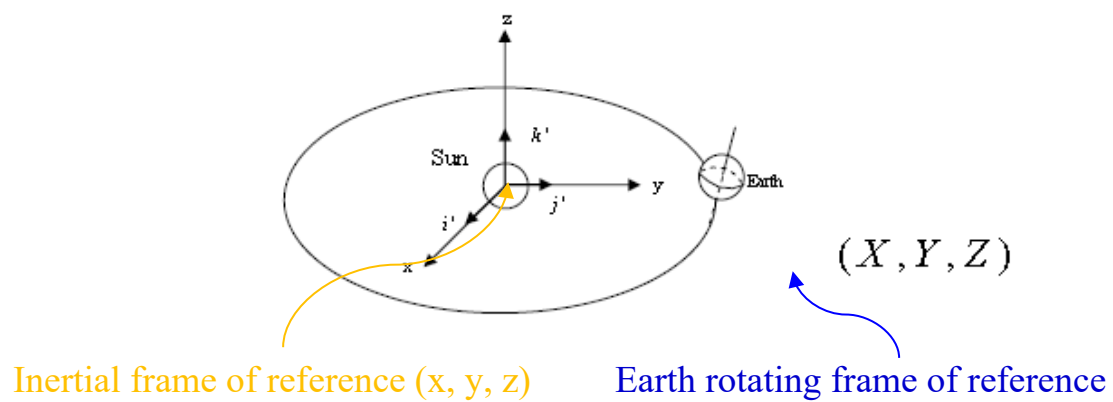
(c) Coordinate systems to be considered

An **inertial (absolute) frame of reference** is a space-time coordinate system that neither rotates nor accelerates. In real life, such a frame of reference is purely theoretical, because gravitational force (and thus acceleration) exists everywhere in the known universe. However, they may be approximated very well in intergalactic space, or to a lesser extent within the confines of a coasting spacecraft.

- For convenience, let us assume that the solar system is an inertial frame of reference (which, in fact, is moving at a speed of ~ 250 km/s around the Milky Way Galaxy – see figure below)
(<http://www.enchantedlearning.com/subjects/astronomy/planets/earth/Speeds.shtml>)



- The Earth's Rotating Frame of Reference



Earth spherical coordinates (λ, ϕ, r) Local coordinates (x, y, z)