

Lecture 15 The Equation of State for Moist Air

(Ref: Sec.4.3 of Hess, Equation of State for Moist Air)

A moist air can be treated as a mixture of dry air and water vapor, i.e.

Moist air = dry air + water vapor

We have already learned about how to compute the mean molecular weight of dry air (M_d) from Lecture 2,

$$\bar{M} = \frac{m_1+m_2+m_3+\dots}{\frac{m_1}{M_1}+\frac{m_2}{M_2}+\frac{m_3}{M_3}+\dots} = \frac{1}{(m_1/m)/M_1+(m_2/m)/M_2+(m_3/m)/M_3+\dots}$$

$$\text{or } \bar{M} = 1 / \sum \left(\frac{m_i / m}{M_i} \right), \quad (2.16)$$

$$\begin{aligned} M_d &= 1 / [(m_{N_2}/m)/M_{N_2} + (m_{O_2}/m)/M_{O_2} + (m_{Ar}/m)/M_{Ar}] \\ &= 1 / [0.7551/28 + 0.2314/32 + 0.0135/40] = 28.97 \text{ kg kmol}^{-1} \end{aligned}$$

Thus, the mean molecular weight of the moist air (\bar{M}) can be computed as follows:

The molecular weights of M_d (dry air) and M_v (water vapor)

$$M_d = 28.97 \text{ kg kmol}^{-1}, M_v = 18 \text{ kg kmol}^{-1},$$

and the respective masses of dry air and water vapor of a moist air, m_d and m_v are used to compute the **mean molecular weight of moist air**

$$\bar{M} = \frac{m_d + m_v}{m_d / M_d + m_v / M_v} .$$

or in terms of the **mass ratio** m_v/m_d , which is usually referred to as the **mixing ratio** ($w=m_v/m_d$):

$$\bar{M} = \frac{1 + w}{1 / M_d + w / M_v} .$$

If we denote M_v/M_d as ε ($=18/28.97=0.622$), then \bar{M} can be rewritten as

$$\bar{M} = \frac{(1 + w)M_d}{1 + w / \varepsilon} .$$

We can further compute the gas constant for the moist air,

$$\bar{R} = \frac{R^*}{\bar{M}} = \frac{(1 + w / \varepsilon)(R^* / M_d)}{1 + w} = \frac{(1 + w / \varepsilon)R_d}{1 + w} .$$

Thus, the equation of state for a moist air may be written,

$$p\alpha = \bar{R}T = R_d \left(\frac{1 + w / \varepsilon}{1 + w} \right) T .$$

If we define a new temperature variable,

$$T_v = \left(\frac{1 + w/\varepsilon}{1 + w} \right) T, \quad (5.3.1)$$

which is called [virtual temperature](#), then the equation of state for a moist air may be expressed in a similar way as the equation of state for a dry air

$$p\alpha = R_d T_v. \quad (5.3.2)$$

The [physical meaning of the virtual temperature](#) is the temperature that the dry air would have if its pressure and specific volume were equal to those of a given sample of moist air.

Since $\varepsilon = M_v / M_d = 18 / 28.97 = 0.622$, and practically $w \ll 1$, Eq. (5.3.1) may be approximated as

$$\begin{aligned} T_v &= \left(\frac{1 + w/\varepsilon}{1 + w} \right) T \approx (1 + w/\varepsilon)(1 - w)T \approx \\ &(1 + (1/\varepsilon - 1)w - w^2/\varepsilon)T \approx (1 + 0.61w)T \\ T_v &= (1 + 0.61w)T \end{aligned} \quad (5.3.3)$$

Next, we need to express w in terms of state variables and known constants, so that it can be measured in the atmosphere.

By definition,

$$w = m_v / m_d = \rho_v / \rho_d \quad (5.3.4)$$

Using the pressures from the equation of state,

$$p_d = \rho_d (R^* / M_d) T,$$

$$e = p_v = \rho_v (R^* / M_v) T,$$

to replace the densities, we obtain,

$$w = \left(\frac{M_v}{M_d} \right) \left(\frac{e}{p_d} \right),$$

or

$$w = \frac{M_v}{M_d} \left(\frac{e}{p - e} \right) \approx \frac{M_v}{M_d} \left(\frac{e}{p} \right) = \varepsilon \left(\frac{e}{p} \right) = 0.622 \frac{e}{p}. \quad (5.3.5)$$

Since w is usually less than 0.04, as a result, $T_v - T < 7 \text{ K}$.

Since the equation of state of dry air is

$$p\alpha = R_d T,$$

comparing with (5.3.2), we may conclude that **dry air is denser than moist air**. This also explains why the moist air rises in the atmosphere.