Orographic Precipitating Systems

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(Ref.: *Mesoscale Dynamics*, Y.-L. Lin, Cambridge, 2007)

Chapter 4. Pure Gravity Waves

(Based on Sec. 3.5 of "Mesoscale Dynamics" by Y.-L. Lin)

- For small-amplitude (linear) perturbations in a 2D ($\partial/\partial y = 0$), inviscid, nonrotating, adiabatic, Boussinesq, uniform basic state flow with uniform stratification, the governing Eqs. (2.2.14) – (2.2.18)

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + V\frac{\partial u'}{\partial y} + U_z w' - fv' + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial x} = 0, \qquad (2.2.14)$$

$$\frac{\partial v'}{\partial t} + U\frac{\partial v'}{\partial x} + V\frac{\partial v'}{\partial y} + V_z w' + fu' + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial y} = 0, \qquad (2.2.15)$$

$$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} + V\frac{\partial w'}{\partial y} - g\frac{\theta'}{\bar{\theta}} + \frac{1}{\bar{\rho}}\frac{\partial p'}{\partial z} + \frac{p'}{\bar{\rho}H} = 0, \qquad (2.2.16)$$

$$\frac{1}{c_s^2}\left(\frac{\partial p'}{\partial t} + U\frac{\partial p'}{\partial x} + V\frac{\partial p'}{\partial y}\right) - \frac{\bar{\rho}}{H}w' + \bar{\rho}\nabla\cdot\boldsymbol{V}' = \frac{\bar{\rho}}{c_p\bar{T}}q', \qquad (2.2.17)$$

$$\frac{\partial \theta'}{\partial t} + U\frac{\partial \theta'}{\partial x} + V\frac{\partial \theta'}{\partial y} + \frac{N^2\bar{\theta}}{g}w' = \frac{\bar{\theta}}{c_p\bar{T}}q', \qquad (2.2.18)$$

reduce to

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + \frac{1}{\rho_o}\frac{\partial p'}{\partial x} = 0, \qquad (3.5.1)$$

$$\frac{\partial w'}{\partial t} + U\frac{\partial w'}{\partial x} - g\frac{\theta'}{\theta_o} + \frac{1}{\rho_o}\frac{\partial p'}{\partial z} = 0, \qquad (3.5.2)$$

$$\frac{\partial u'}{\partial x}+\frac{\partial w'}{\partial z}=0, \qquad (3.5.3)$$

$$\frac{\partial \theta'}{\partial t}+U\frac{\partial \theta'}{\partial x}+\frac{N^2\theta_o}{g}w'=0, \qquad (3.5.4)$$

where θ_o: a constant reference θ (potential temperature)

$N^2[\equiv(g/\theta_o)\partial\bar{\theta}/\partial z]$: square of the Boussinesq Brunt-Vaisala (buoyancy) frequency.

Note: For a two-dimensional, nonrotating fluid flow, there is no need to retain the meridional (*y*-) momentum equation in our system of equations, because *v'* will keep its initial value for all time, as required by the reduced form of the *y*-momentum equation, namely, $\partial v'/\partial t+U\partial v'/\partial x=0$.

However, the *y*-momentum equation needs to be kept if the fluid is two-dimensional and rotating ($f\neq 0$) since the initial v' will vary with time, although independent of *y*, due to the presence of Coriolis force.

Figure 3.7 illustrates the vertical oscillation of an air parcel in a stratified atmosphere with a Brunt-Vaisala frequency *N*. The total oscillation period is $2\pi/N$ (τ_b in the figure).

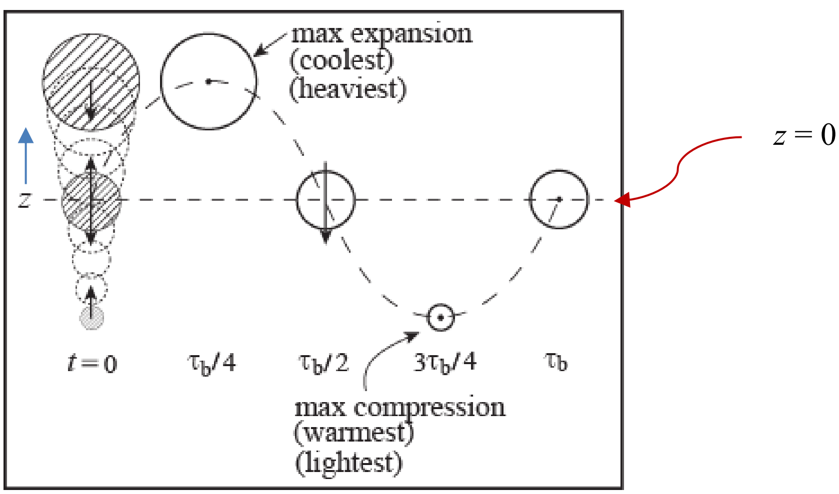

Fig. 3.7: Vertical oscillation of an air parcel in a stably stratified atmosphere when the Brunt-Vaisala frequency is *N*. The oscillation period of the air parcel is $\tau_b=2\pi/N$ and the volume of the air parcel is proportional to the area of the circle. (Adapted after Hooke 1986; quoted in Lin 2007)

- The air parcel goes through the following processes:
  - (a) The air parcel expands and cools while it ascends, and reaches its maximum expansion and coolest state (highest ρ') at $t = \tau_b/4$.
  - (b) It then starts to descend back to its original level due to negative buoyancy, which overshoots passing its original level ($z = 0$) at $t = \tau_b/2$.
  - (c) The air parcel compresses and warms adiabatically while it descends passing the original level ($z = 0$), and reaches its maximum compression and warmest state at $t = 3\tau_b/4$.
  - (d) At this level (at $t = 3\tau_b/4$), the air parcel reaches its lowest ρ', which then ascends due to positive buoyancy, and returns to its original level at $t = \tau_b$.

- Equations (3.5.1) – (3.5.4) may be combined into a single equation for the vertical velocity w', which is a simplified form of the *Taylor-Goldstein equation* [(3.7.19)] in the absence of vertical wind shear,

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial z^2}\right)+N^2\frac{\partial^2 w'}{\partial x^2}=0. \qquad (3.5.5)$$

- Assuming a traveling sinusoidal plane wave solution of the form,

$$w' = \hat{w}(z)\; e^{i(kx-\omega t)}, \qquad (3.5.6)$$

and substituting it into (3.5.5) yields the following linear partial differential equation with constant coefficients, which governs the vertical structure of w',

$$\frac{\partial^2 \hat{w}}{\partial z^2}+\left(\frac{N^2}{\Omega^2}-1\right)k^2\hat{w}=0. \qquad (3.5.7)$$

In the above equation, $\Omega \equiv \omega - kU$ is the *intrinsic* (*Doppler-shifted*) *frequency* of the wave relative to the uniform basic state flow.

[How to solve a homogeneous 2nd order ODE with constant coefficients: Eq. (3.5.7) is analogous to a general ODE: $y'' + py' + qy = 0$, $y' = dy/dx$, which has a characteristic (auxiliary) equation: $k^2 + pk + q = 0$ having 2 roots, k_1 & k_2, and a solution of $y(x) = C_1e^{k_1x} + C_2e^{k_2x}$.]

- Equation (3.5.7) has the following two solutions:

$$\hat{w} = A e^{ik\sqrt{N^2/\Omega^2 - 1}\, z} + B e^{-ik\sqrt{N^2/\Omega^2 - 1}\, z}, \quad \text{for } N^2/\Omega^2 > 1, \tag{3.5.8}$$

and

$$\hat{w} = C e^{k\sqrt{1 - N^2/\Omega^2}\, z} + D e^{-k\sqrt{1 - N^2/\Omega^2}\, z}, \quad \text{for } N^2/\Omega^2 < 1. \tag{3.5.9}$$

- Note that the above 2 equations may also be written together as

$$\hat{w} = A e^{imz} + B e^{-imz}$$

where *m*, the vertical wave number, is defined as

$$m^2 = k^2 (N^2/\Omega^2 - 1)$$

Equation (3.5.8) represents a vertically propagating wave because it is sinusoidal with height. As will be discussed in Section 4.4, term *A* represents a wave with upward energy propagation, while term *B* represents a wave with downward energy propagation.

Thus, for waves generated by orography, term *B* is unphysical and has to be removed because the wave energy source is located at the surface, as required by the *radiation boundary condition*.

- On the other hand, term *C* of (3.5.9) represents wave amplitude increasing exponentially with height, while term *D* represents a wave whose amplitude decreases exponentially from the level of wave generation.

Thus, for waves or disturbances generated by orography, term *C* is unphysical. This is also called the *boundedness condition*. Under this situation, term *D* represents an *evanescent wave* (*disturbance*), whose wave amplitude decreases exponentially with height.

- In other words, there exist two distinct flow regimes for pure gravity waves (i.e. *vertically propagating waves* and *evanescent waves*) in the atmosphere, which are determined respectively by the following criteria:

$$N^2/\Omega^2 > 1 \quad \text{and} \quad N^2/\Omega^2 < 1. \tag{3.5.10}$$

- The above two pure gravity wave flow regimes can be understood by considering steady state responses of stably stratified airflow over a sinusoidal topography.

  When $N^2/\Omega^2 > 1$, we have $2\pi/N < L/U$, where $L = 2\pi/k$ is the dominant horizontal wavelength of the sinusoidal topography.

  Note that $2\pi/N$ is the buoyancy oscillation period and that L/U is the advection time an air parcel takes to cross over the mountain. Thus, fluid particles take less time to oscillate in the vertical, compared to the horizontal advection time required to pass over the mountain. This allows the wave energy to propagate vertically (Fig. 3.8a).

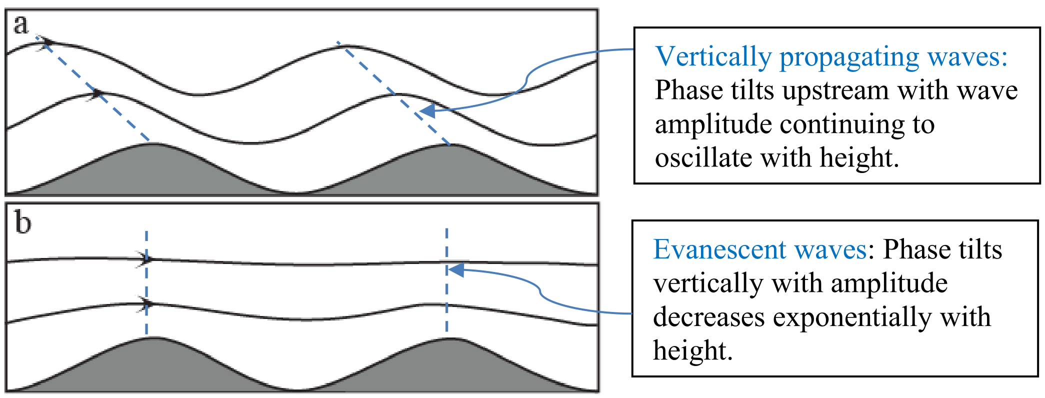

Fig. 3.8: (a) Vertically propagating waves ($N^2/\Omega^2 > 1$) and (b) evanescent waves ($N^2/\Omega^2 < 1$) for a linear, two-dimensional, inviscid flow over sinusoidal topography.

- On the other hand, when $N^2/\Omega^2 < 1$ or $2\pi/N > L/U$, fluid particles do not have enough time to oscillate vertically because the time required for the particles to be advected over the mountain is shorter.

  Therefore, the wave energy cannot freely propagate vertically, and it is preferentially advected downstream, remaining near the Earth's surface (Fig. 3.8b). This type of wave or disturbance is also referred to as an evanescent wave or a *surface trapped wave*.

- If the stratification of the fluid is uniform (N = constant) and the disturbance is sinusoidal in the vertical, then $\hat{w}$ may be written as $\hat{w} = w_o e^{imz}$, where w_o and m are the wave amplitude and vertical wave number, respectively. Substituting $\hat{w}$ into (3.5.7)

$$\frac{\partial^2 \hat{w}}{\partial z^2} + \left(\frac{N^2}{\Omega^2} - 1 \right) k^2 \hat{w} = 0 . \qquad (3.5.7)$$

yields the dispersion relationship for pure gravity waves,

$$\Omega = \frac{\pm\ Nk}{\sqrt{k^2 + m^2}} . \qquad (3.5.11)$$

- Returning to the vertical structure solutions, (3.5.8) and (3.5.9), there are two extreme cases that merit further discussion.

$$\hat{w} = A\, e^{ik\sqrt{N^2/\Omega^2 - 1}\, z} + B\, e^{-ik\sqrt{N^2/\Omega^2 - 1}\, z}, \quad \text{for } N^2/\Omega^2 > 1 , \qquad (3.5.8)$$

and

$$\hat{w} = C\, e^{k\sqrt{1 - N^2/\Omega^2}\, z} + D\, e^{-k\sqrt{1 - N^2/\Omega^2}\, z}, \quad \text{for } N^2/\Omega^2 < 1 . \qquad (3.5.9)$$

When $N^2 >> \Omega^2$, the buoyancy oscillation period ($2\pi / N$) is much shorter than the oscillation period of the disturbance ($2\pi / \omega$) or the advection time (L/U).

Therefore, the wave energy will propagate purely in the vertical direction. In this situation ($N^2 >> \Omega^2$), constant phase lines and group velocities are oriented vertically, while the total wave number vector is oriented horizontally.

In this special flow regime, often referred to as the *hydrostatic gravity wave regime*, the vertical momentum equation (3.5.2)

$$\frac{\partial w'}{\partial t} + U \frac{\partial w'}{\partial x} - g \frac{\theta'}{\theta_o} + \frac{1}{\rho_o} \frac{\partial p'}{\partial z} = 0 , \qquad (3.5.2)$$

reduces to its hydrostatic form,

$$\frac{1}{\rho_o} \frac{\partial p'}{\partial z} = g \frac{\theta'}{\theta_o} . \qquad (3.5.18)$$

- This implies that the vertical pressure gradient force is in balance with the buoyancy force in the *z* direction. In other words, vertical acceleration Dw'/Dt

plays an insignificant role in wave propagation. It can be shown from (3.5.8) that the waves repeat themselves in the vertical direction without losing their amplitude and have a wavelength of $2\pi\Omega / kN$ for a steady state flow. For hydrostatic gravity waves, the wave equation (3.5.5)

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial z^2}\right)+N^2\frac{\partial^2 w'}{\partial x^2}=0\,. \qquad (3.5.5)$$

for the vertical velocity w' reduces to

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\frac{\partial^2 w'}{\partial z^2}+N^2\frac{\partial^2 w'}{\partial x^2}=0\,. \qquad (3.5.19)$$

Note that by assuming $\partial^2 w'/\partial x^2 << \partial^2 w'/\partial z^2$, it implies that the horizontal scale of the motion is much larger than the vertical scale, i.e. $L_x >> L_z$. In other words, the motion is shallow. In general, hydrostatic assumption applies to a fluid system or motion which is shallow.

- In the other limit, $N^2 << \Omega^2$, the buoyancy oscillation period is much greater than that of the disturbance ($2\pi/\omega$) or advection time of the air parcel (L/U).

  Therefore, the buoyancy force plays insignificant role in this flow regime. In this situation ($N^2 << \Omega^2$), the wave energy is not able to propagate vertically, and the wave disturbance will remain locally in the vicinity of the forcing.

  The vertical momentum equation, (3.5.2),

$$\frac{\partial w'}{\partial t}+U\frac{\partial w'}{\partial x}-g\frac{\theta'}{\theta_o}+\frac{1}{\rho_o}\frac{\partial p'}{\partial z}=0\,, \qquad (3.5.2)$$

  reduces to

$$\frac{\partial w'}{\partial t}+U\frac{\partial w'}{\partial x}=-\frac{1}{\rho_o}\frac{\partial p'}{\partial z}\,. \qquad (3.5.20)$$

Thus, only the vertical pressure gradient force contributes to the vertical acceleration. It can also be shown from (3.5.9) that the amplitude of the disturbance decreases exponentially with height. As discussed earlier, this special case is called the *evanescent flow regime*.

The wave equation for w' then reduces to (for evanescent flow regime)

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)^2\left(\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial z^2}\right)=0. \tag{3.5.21}$$

If the flow starts with no relative vorticity in the y-direction (i.e. if $\partial u'/\partial z-\partial w'/\partial x=0$ at t = 0), then the above equation reduces to a two-dimensional form of the Laplace's equation

$$\frac{\partial^2 w'}{\partial x^2}+\frac{\partial^2 w'}{\partial z^2}=0. \tag{3.5.22}$$

Because this type of flow is everywhere vorticity-free, it is often referred to as *potential* (*irrotational*) *flow*.

[More discussions on wave properties] – reading assignment

- For a quiescent fluid, $\Omega = \omega$ ($\because \Omega = \omega - kU$ & U = 0), Eq. (3.5.11) reduces to

$$\omega = \frac{\pm Nk}{\sqrt{k^2+m^2}}, \tag{3.5.12}$$

or

$$\frac{\omega}{N} = \frac{\pm k}{\sqrt{k^2+m^2}} = \pm\cos\alpha, \tag{3.5.13}$$

where α is the angle ($|\alpha| \le \pi/2$) between the wave number vector $\boldsymbol{k}$ = (k, m) and the x-axis.

- First consider the extreme case of N = 0 (no stratification => no vertical oscillation, i.e. no gravity waves: $kx - \omega t = 0$ => $c_p = x/t = \omega/k$. The wave generator in a quiescent, homogeneous fluid will send water waves out horizontally.

Then allow the stratification (N) to occur and increase => generate vertical oscillations, which will send 4 rays out (see water tank experiment in Mowbray and Rarity 1967).

Note that fluid parcels oscillate in a direction perpendicular to the total wave number vector, as indicated by the incompressible continuity equation, $\boldsymbol{k} \cdot \boldsymbol{V}' = 0$. Therefore, the wave fronts or rays associated with particle oscillations tilt at an angle α with respect to the vertical. For a given stratification, waves with constant $\omega < N$ propagate at a fixed angle to the horizontal axis, which is independent of the wavelength.

The above characteristics can be illustrated in Fig. 3.9. While the wave number vector is oriented in the same direction as the phase speed vector [$\boldsymbol{c_p}$ in Fig. 3.9], the wave front is oriented perpendicular.

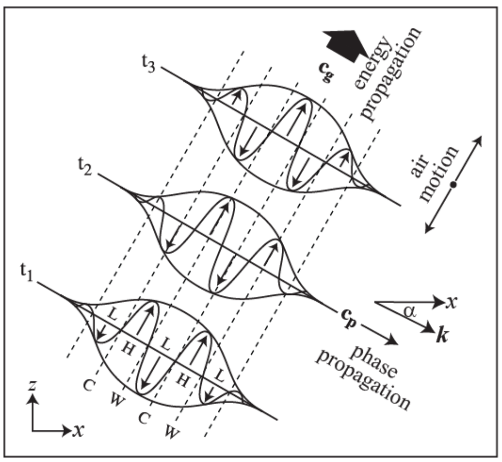

Fig. 3.9: Basic properties of a vertically propagating gravity wave with $k > 0, m < 0,$ and $\omega > 0$. The energy of the wave group propagates with the group velocity ($\boldsymbol{c_g}$, thick blunt arrow), while the phase of the wave propagates with the phase speed ($\boldsymbol{c_p}$). Relations between w', u', p', and θ' as expressed by (3.5.16) and (3.5.17) are also sketched. Symbols H and L denote the perturbation high and low pressures, respectively, while W and C denote the warmest and coldest regions, respectively, for the wave at t_1. Symbol α defined in (3.5.13) represents the angle of the wave number vector $\boldsymbol{k}$ from the horizontal axis or the wave front (line of constant phase) from the vertical axis. (Adapted after Hooke 1986)

The wave is dispersive when:

(1) $c_p = c_g$ or

(2) c_p is not a function of wave number (i.e. wavelength)

Dispersion

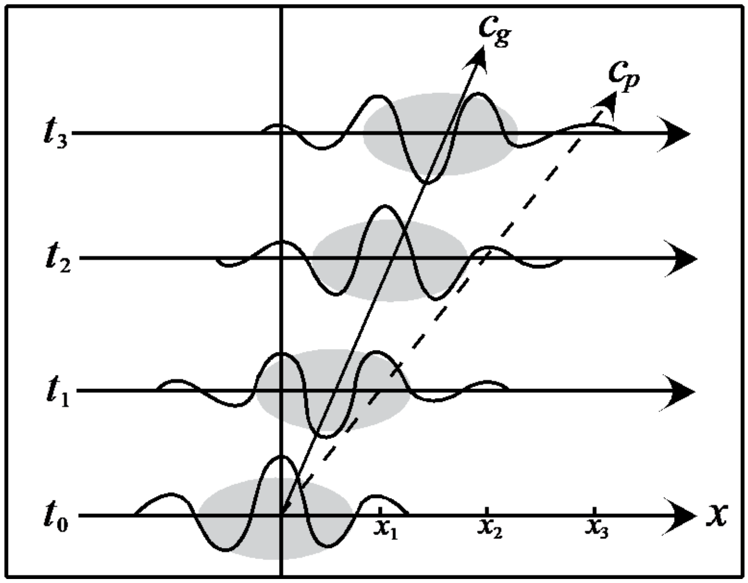

Fig. 3.1: Propagation of a wave group and an individual wave. The solid and dashed lines denote the group velocity (c_g) and phase velocity (c_p), respectively. Shaded oval denotes the concentration of wave energy which propagates with the group velocity. The phase speed c_p equals x_i / t_i, where i = 1, 2, or 3.

- **[Application to the lateral boundary conditions in a numerical model]**

From (3.5.12), we may obtain the horizontal and vertical phase velocities,

$$c_{px}=\frac{\omega}{k}=\frac{\pm N}{\sqrt{k^2+m^2}};$$
$$c_{pz}=\frac{\omega}{m}=\frac{\pm kN}{m\sqrt{k^2+m^2}} \qquad (3.5.14)$$

These expressions indicate that pure gravity waves are dispersive in both the x and z directions because both c_{px} and c_{pz} depend on wave number. The group velocities can be derived from (3.5.12),

$$c_{gx}=\frac{\partial\omega}{\partial k}=\frac{\pm m^2N}{\left(k^2+m^2\right)^{3/2}}; \qquad c_{gz}=\frac{\partial\omega}{\partial m}=\frac{\mp kmN}{\left(k^2+m^2\right)^{3/2}}. \qquad (3.5.15)$$

Note that c_{px} and c_{gx} are directed in the same direction, while c_{pz} and c_{gz} are directed in opposite directions. This is also shown in Fig. 3.9.

Due to these peculiar properties of internal gravity waves, the implementations for lateral and upper boundary conditions associated with mesoscale numerical models that resolve these waves must be carefully configured.

- Briefly speaking, a horizontal advection equation, $\partial\varphi/\partial t+c_{px}\partial\varphi/\partial x=0$, where φ represents any prognostic dependent variable, can be applied at the lateral boundaries and can be implemented to help advect the wave energy out of the lateral boundary of the computational domain.
- On the other hand, a vertical advection equation, $\partial\varphi/\partial t+c_{pz}\partial\varphi/\partial z=0$ (with $c_{pz}>0$), cannot advect the wave energy out of the upper boundary since the wave energy will propagate downward back into the computational domain as c_{gz} is negative. The numerical radiation boundary conditions will be discussed in more detail in Section 13.2, while the details of the Sommerfeld (1949) radiation boundary condition will be discussed in Section 4.4 (Lin 2007).

Due to the fact that only the real part of the solution is physical, (3.5.6)

$$w' = \hat{w}(z)\ e^{i(kx-\omega t)}, \tag{3.5.6}$$

and $\hat{w}(z) = w_o \exp(imz)$ can be combined in the form,

$$w' = \mathrm{Re}\left(w_o e^{i(kx+mz-\omega t)}\right) = w_r \cos(kx+mz-\omega t) - w_i \sin(kx+mz-\omega t),$$

$$\theta' = \left(\theta_o N^2/g\omega\right)\left[w_r \sin(kx+mz-\omega t) + w_i \cos(kx+mz-\omega t)\right] \tag{3.5.16}$$

where w_r and w_i are the real and imaginary parts of w_o, respectively.

- Substituting w' into (3.5.1) – (3.5.4) and assuming $U = 0$ leads to the *polarization relations*

$$u' = -(m/k)\left[w_r \cos(kx+mz-\omega t) - w_i \sin(kx+mz-\omega t)\right], \tag{3.5.17a}$$

$$p' = -\left(\rho_o \omega m/k^2\right)\left[w_r \cos(kx+mz-\omega t) - w_i \sin(kx+mz-\omega t)\right], \tag{3.5.17b}$$

$$\theta' = \left(\theta_o N^2/g\omega\right)\left[w_r \sin(kx+mz-\omega t) + w_i \cos(kx+mz-\omega t)\right]. \tag{3.5.17c}$$

The above relationships are also shown in Fig. 3.9 for the case where $k > 0$, $m < 0$, and $\omega > 0$. The wave frequency is assumed to be positive, in order to avoid redundant solutions.

For $k > 0$, $m < 0$, and $\omega > 0$, (3.5.17a) indicates that u' is in phase with w', which is shown in Fig. 3.9 by fluid oscillating toward the right in regions of upward motion.

Equation (3.5.17b) indicates that p' is also in phase with w'. Thus, high (low) pressure is produced in regions of upward (downward) motion.

Equation (3.5.17c) indicates that θ' is out of phase with w' by $\pi/2$ (90°). A fluid particle loses (gains) buoyancy in regions of upward (downward) motion, according to (3.5.4) with $U = 0$. Therefore, the least buoyant (coldest) fluid parcels (denoted by C in t_1 of Fig. 3.9) will move toward regions of maximum upward motion. That is, internal gravity waves will move in the direction of phase propagation (toward the lower right corner of the figure), as denoted by $\boldsymbol{c}_p$ in the figure.