ASME 231 **Atmospheric Thermodynamics**

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Lecture 21 Hydrostatic Equilibrium

(Ch.6 of Hess)

7.1 The hydrostatic equation

Because thermodynamic principles are often applied to parcels of air moving in the vertical, it is useful to include the equation describing the balance of forces in the vertical direction in the atmosphere.

Newton's second law of motion states:

$$F = ma$$

Consider a slab of air with thickness Δz, horizontal area A and mass m.

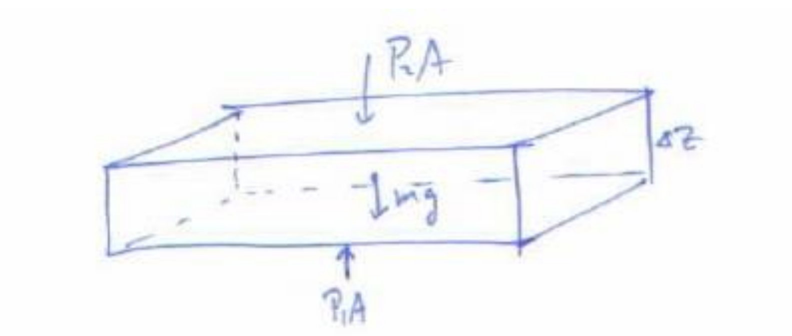


Fig. 21.1

Vertical velocity = $w = dz/dt$

Vertical acceleration: $a = dw/dt = d^2z/dz^2$

Gravitational force: $G = -mg$
Pressure gradient force in the vertical:

$$PGF = p_1 A - p_2 A = (p_1 - p_2)A$$

Therefore, Newton's second law can be written for this slab of air as

$$F = PGF + G = (p_1 A - p_2)A - mg = (p_1 - p_2)A - mg$$

$$ma = m\,dw/dt$$

$$(p_1 - p_2)A - mg = m\,dw/dt\,.$$

For small Δz, we may approximate p_2 by

$$p_2 = p_1 + \frac{\partial p}{\partial z}\Delta z,$$

or

$$p_1 - p_2 = -\frac{\partial p}{\partial z}\Delta z.$$

Substituting the above equation into the previous one leads to

$$\frac{dw}{dt} = -\frac{\partial p}{\partial z}\left(\frac{A\Delta z}{m}\right) - g = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g\,.$$

When the gravitational force (-*mg*) exactly balances the pressure gradient force in vertical, then there exist no vertical acceleration. That means the atmosphere is in hydrostatic balance:

$$\frac{\partial p}{\partial z} = -\rho g \; .$$

If p is treated as a function of z only, then it can be written as

$$\frac{dp}{dz} = -\rho g \; .$$

7.2 Geopotential

As defined earlier in the course, geopotential is the work that must be done to raise 1 kg of mass from sea level to some height z against gravity. It has a unit of $J\,kg^{-1}$ or $m^2 s^{-2}$.

In other words, it is the amount of work done by gravitational force on 1 *kg* of mass during the entire lifting process. Note that work can be calculated by multiplying the distance and the force. Thus, the change of potential energy for lifting an air parcel for a distance dz against gravity is

$$d\phi = g\, dz \, .$$

The total potential energy (work done against gravity) is

$$\phi(z) = \int_0^z g dz \, ,$$

where ϕ depends on the height, not the path. The surface at which the geopotential is a constant is called an *equi-geopotential surface*. Note that it is not a surface of constant elevation since g is not a constant.

In order to solve this problem, we introduce a new variable, geopotential height (*Z*), which is defined as

$$Z = \frac{\phi(z)}{g_o} = \frac{1}{g_o}\int_0^z g\,dz\,,$$

where $g_o = 9.8\ ms^{-2}$ is the globally averaged acceleration due to gravity at the earth's surface.

In the lower atmosphere, $g \sim g_o$, so $Z \sim z$.

Example: A pressure of 700 mb is found on a certain day to be at an "elevation" of 3000 geopotential meters. What is the elevation in geometrical meters? Assume the local gravity is g=9.806.

Answer:

$3000\ geoM = 1/9.8 \times 9.806\ z = 9.806\ z/9.8$
$z = 2998\ m.$
$Z - z = 2\ m.$

An approximate formula for gravity is:

$$g(z) = g_o\left(1 - \frac{2z}{a}\right) = g_o\left(1 - 3.14x10^{-7} z\right)$$

where g_o is the gravitational acceleration at the sea level.

In meteorological applications, it is not convenient to deal with density. So, we use the equation of state to rewrite the hydrostatic equation

$$\frac{\partial p}{\partial z} = -\rho g = -\frac{pg}{RT} = -\frac{pg}{R_d T_v}.$$

Accordingly,

$$d\phi = g\ dz = -\frac{R_d T_v dp}{p} = -R_d T_v\ d(\ln p).$$

Thus,

$$\phi_2 - \phi_1 = \int_1^2 d\phi = \int_{p1}^{p2} -R_d T_v d(\ln p) = -R_d \int_{p1}^{p2} T_v d(\ln p).$$

By the same token, the geopotential height difference is may be calculated by

$$z_2 - z_1 = -\frac{R_d}{g_o} \int_{p1}^{p2} T_v d(\ln p).$$

7.3 Scale height and hypsometric equation

Radiosonde or rawinsonde measure p, T, and humidity. One basic problem is to determine the elevation from these measurements.

For an isothermal (i.e., T is constant with height) atmosphere, we have

$$p_2 = p_1\, e^{-(Z_2-Z_1)g_o/R_dT_v} = p_1\, e^{-(Z_2-Z_1)/H}.$$

Let $p_1=p_o$, $p_2=p$, $Z_1=0$, and $Z_2=Z$, then we have

$$p = p_o\, e^{-Zg_o/R_dT_v} = p_o\, e^{-Z/H}.$$

where $H = R_dT_v/g_o$ is called the *scale height* of the atmosphere. It is the e-folding height of the sea level pressure.

Z	p/p0
0	1
H	e^{-1}
2H	e^{-2}
3H	e^{-3}

i.e., the pressure decreases by a factor of e (=2.718) for every increase of H in geopotential height.

If we take $T_v = 273^{\circ}K$, then H = 287 x 273 / 9.8 = 8 km.

Since R depends on the molecular weight, p(z) depends on the composition of the atmosphere.

From the equation of state, it is easy to derive

$$\rho = \rho_o \exp(-Z/H).$$

That is, density and pressure have the same vertical distribution.

In the real atmosphere, temperature varies with height. However, we can define a mean virtual temperature

$$\bar{T}_v = \int_{p1}^{p2} T_v \, d(\ln p) \Big/ \int_{p1}^{p2} d(\ln p) = \left(\int_{p1}^{p2} T_v d\ln p\right) / \ln(p_1/p_2)$$

then

$$Z_2 - Z_1 = \frac{R_d}{g_o} \ln\left(\frac{p_1}{p_2}\right).$$

This equation is called the *hypsometric equation*.

Applications:

1) Thickness and heights of constant pressure surfaces

$Z_2 - Z_1$: is the difference or thickness in geopotential height between 2 pressure levels. It is proportional to the mean virtural temperature between the two pressure levels. Thus, the layer becomes thicker as the mean virtual temperature increases.

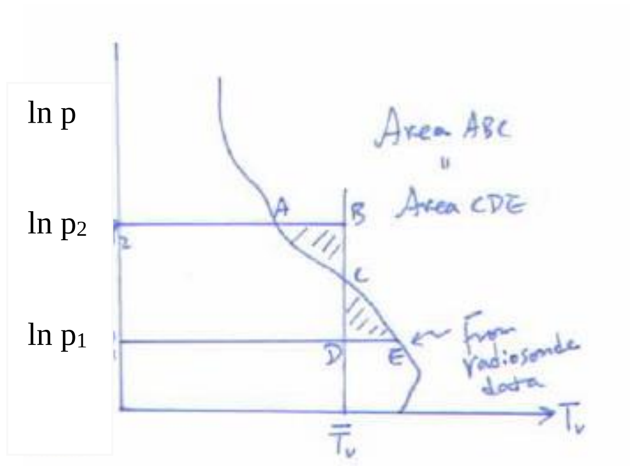
Example: Calculate the thickness of the layer between 1000 and 500 mb pressure levels: (a) in the tropics where $T_V = 9^{\circ}C$ and (b) in the polar region where $T_V = -40^{\circ}C$.

Answer: $\Delta Z = Z_{500} - Z_{1000mb} = R_d T_V / g_0 \ln(1000/500) = 20.3\ T_V$

(a) $T_V = 282^{\circ}K$ (such as in the tropics), $\Delta Z = 5725$ m

(b) $T_V = 233^{\circ}K$ (such as in the polar region), $\Delta Z = 4730$ m.

Use this method, we can construct topographical maps of Z for every pressure level.



i) Radiosonde data provides p, T, T_d.

ii) Calculate T_V at each level at each station.

iii) Plot on a thermodynamic diagram.

iv) Find mean virtual temperature T_V for the layer.

v) Calculate ΔZ from T_V for each station.

vi) Plot contours of ΔZ from a network of stations.

Thus, if T_V (x,y,z) and Z_0 are known on p_0 surface, Z(x,y, z) can be found. This type of map is useful in giving insights into the 3-D structure of atmospheric disturbances, e.g. fronts.

Examples:

a) Intensity of a warm low must decrease with height.

b) "cold core" upper tropospheric low in mid-latitude can not extend to surface.

c) mid-latitude cyclones tilt westward.